Reach Set Computation and Control Synthesis for Discrete-Time Dynamical Systems with Disturbances

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Abstract

Control problems with hard bounds on the control values, restrictions on the state trajectory over a finite time horizon, and guaranteed behavior despite the disturbances, are difficult to solve using frequency based design methods. Such problems have received much attention over the past decade. In order to address them one needs to study system evolution in the time domain. The central concept that emerges in such studies is that of the *reach* set. This paper is devoted to the formulation of the reachability problem for discrete-time dynamical systems with disturbances. The concept of maxmin and minmax forward and backward reach sets is addressed. Invariance of the backward reach set is discussed. The emphasis of the paper is on discrete-time linear systems, for which the ellipsoidal computational method is described. The synthesis of maxmin and minmax closed-loop control for steering the system to a given target set using ellipsoidal backward reach set approximations is explained. The ellipsoidal method covered in the paper is implemented in the Ellipsoidal Toolbox for MATLAB, a popular collection of ellipsoidal calculus routines freely available online.

1 Introduction

Traditional control theory is concerned with the design of linear feedback control with desirable asymptotic behavior, such as stability and small steady state and tracking errors, while properties of transient behavior are expressed in terms of overshoot and speed of response. External disturbances can be handled by modeling these as random processes, leading to the Linear Quadratic Gaussian (LQG) problem formulation. This theory has some limitations.

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Because the feedback law is specified to be linear, it is not possible to explicitly incorporate hard bounds on the control values, e.g., the requirement that the applied force should not exceed a specified limit. Second, it is not possible to express finite time requirements, e.g., the requirement that the system state reach a pre-specified value at a pre-specified time. Third, it is not possible to demand guaranteed performance in the face of disturbances, e.g, the requirement that a certain target state be reached, no matter what the disturbance.

In order to address these problems one needs to study system evolution in the time domain. The central concept that emerges in such studies is that of the *reach* set, which is the set of states to which the system can be steered using available controls. This paper is devoted to the formulation and computation of the reach set of a discrete-time linear system with disturbances with emphasis. The concept of reachability was introduced in [28]; [29] shows the reach set can be computed by solving the forward Hamilton-Jacobi-Bellman-Isaacs (HJBI) partial differential equation; and the notion of backward reachability with its application to reaching a specified target set is described in [18]. Reachability of hybrid systems is addressed in [34, 31]. Over the last decade, significant advances were made in the characterization of reach sets and their computation for linear systems.

Computation of reach sets as level sets of HJBI solutions was introduced in [20, 19, 22] with special emphasis on linear systems. In [32] the authors applied the level set method to reachability analysis of hybrid systems. The level set method is implemented in the *Level Set Toolbox* [4], which uses numerical algorithms for time-dependent HJBI equations and structured grids.

For some classes of systems the reach sets can be computed symbolically using *quantifier elimination*. In [27] the authors deal with linear time-invariant systems with nilpotent Jacobian matrix. Requiem [8] is a Mathematica notebook which, given a linear system, the set of initial conditions and control bounds, symbolically computes the exact reach set, using the experimental quantifier elimination package.

There are reach set algorithmic computation methods based on general polytopes, as well as those with specific structure. The general polytope method consists in sequential computation of affine transformations of polytopes, geometric sum of two polytopes, and geometric difference of two polytopes. It is implemented in the Multi-Parametric Toolbox (MPT) for MATLAB [26, 6]. Polytopes can give arbitrarily close approximations to any convex set, but the number of vertices can grow prohibitively large and, as shown in [12], the computation of a polytope by its convex hull becomes intractable for large number of vertices in high dimensions. Symmetric polytopes, called *zonotopes* [9], could be a solution. The zonotope method for external approximation of reach sets of discrete-time linear systems was introduced in [15], implemented in the *MATISSE* package for MATLAB [5], and further discussed in [16]. In the d/dt [2] verification tool, the reach set of a discrete-time linear system without disturbance is over-approximated by unions of rectangular polytopes [11]. CheckMate [1] is a MATLAB toolbox that can evaluate specifications for trajectories starting from the set of initial (continuous) states corresponding to the parameter values at the vertices of the parameter set. This provides preliminary insight into whether the specifications will be true for all parameter values. The method of oriented rectangluar polytopes for external approximation of reach sets is introduced in [35]. Currently, the development of CheckMate is discontinued. Therefore, we refer the reader to *PHAVer* [7], the newly developed verification tool.

There are analytic methods for over- and underapproximation of reach sets based on ellipsoids [21, 23, 25] and parallelotopes[1] [17]. Application of ellipsoidal method to hybrid systems is described in [24]. These methods introduce parametrized families of external and internal appreoximating ellipsoids or parallelotopes and provide differential or difference equations that govern the centers and shape matrices of those approximations. Ellipsoidal reachability methods are implemented in the Ellipsoidal Toolbox [3].

For certain verification problems computation of reach sets can be avoided. For example, it may be enough to ensure that for given set of initial conditions, the trajectories of system never enter a given target set. In this case, the method of *barrier certificates* [30, 33] may be useful. Another example for which reach sets need not be computed occurs when it is possible to ensure that for given initial set there exist system trajectories that never leave this initial set. Such an initial set is said to be *invariant* with respect to those trajectories. In [10] the authors show that for certain classes of discrete-time dynamical systems with disturbances and certain initial sets, convex constraints on controls and disturbances, for every disturbance there exist closed-loop control strategies that keep the state of the system inside the initial sets. For more information about invariant sets, we refer the reader to the survey paper [13] and references therein.

A more detailed critical overview of the existing reachability methods and tools can be found in the Ellipsoidal Toolbox manual [3].

The rest of the paper is organized as follows. Section 2 introduces the forward and backward reach sets, the classes of open- and closed-loop controls, and different kinds of reach sets that are appropriate for dealing with disturbances. Although some of the discussion applies to nonlinear systems, explicit formulas for reach sets are available only for linear systems. Geometric operations with ellipsoids that deal with external and internal approximations of ellipsoidal difference-sum and sum-difference are presented in Section 3. Section 4 is devoted to a set of algorithms based on these ellipsoidal operations. Control synthesis for steering the system to a given target set in given time is addressed in 5, covering both, maxmin and minmax cases. Section 6 provides some examples.

2 Reachability analysis

2.1 Systems without disturbances

Consider a general discrete-time dynamical system

$$x(t+1) = f(t, x, u), \tag{2.1}$$

[1] Parallelotope is a zonotope with 2^n vertices, where n is the state space dimension.

wherein t is the time step, $x \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}^m$ is the control, and f is a measurable vector function taking values in $\mathbf{R}^n$. The control values $u(t,x)$ are restricted to a closed compact control set $\mathcal{U}(t) \subset \mathbf{R}^m$. An *open-loop* control does not depend on the state, $u = u(t)$; for a *closed-loop* control, $u = u(t,x)$.

Definition 2.1 (Reach set) *The (forward) reach set $\mathcal{X}(t,t_0,x_0)$ at time step $t > t_0$ from the initial position (t_0,x_0) is the set of all states $x(t)$ reachable at time step t by system (2.1), with $x(t_0) = x_0$, through all possible controls $u(\tau,x) \in \mathcal{U}(\tau)$, $\tau = t_0..(t-1)$. For a given set of initial states $\mathcal{X}_0$, the reach set $\mathcal{X}(t,t_0,\mathcal{X}_0)$ is*

$$\mathcal{X}(t,t_0,\mathcal{X}_0) = \bigcup_{x_0 \in \mathcal{X}_0} \mathcal{X}(t,t_0,x_0).$$

Here are two facts about forward reach sets.

1. $\mathcal{X}(t,t_0,\mathcal{X}_0)$ is *the same* for open-loop and closed-loop control.
2. $\mathcal{X}(t,t_0,\mathcal{X}_0)$ satisfies the semigroup property,
$$\mathcal{X}(t,t_0,\mathcal{X}_0) = \mathcal{X}(t,\tau,\mathcal{X}(\tau,t_0,\mathcal{X}_0)), \quad t_0 \leq \tau < t. \tag{2.2}$$

For linear systems
$$f(t,x,u) = A(t)x(t) + B(t)u, \tag{2.3}$$
with matrices $A(t)$ in $\mathbf{R}^{n\times n}$ and $B(t)$ in $\mathbf{R}^{m\times n}$. The state transition matrix is

$$\Phi(t+1,t_0) = A(t)\Phi(t,t_0), \quad \Phi(t,t) = I,$$

which for constant $A(t) \equiv A$ simplifies as

$$\Phi(t,t_0) = A^{t-t_0}.$$

If the state transition matrix is invertible, $\Phi^{-1}(t,t_0) = \Phi(t_0,t)$. The state transition matrix is always invertible for sampled continuous-time linear systems. However, if for some τ, $t_0 \leq \tau < t$, $A(\tau)$ is degenerate (singular), $\Phi(t,t_0) = \prod_{\tau=t_0}^{t-1} A(\tau)$, is also degenerate and cannot be inverted.

The reach set $\mathcal{X}(t,t_0,\mathcal{X}_0)$ for a linear system can be expressed as

$$\mathcal{X}(t,t_0,\mathcal{X}_0) = \Phi(t,t_0)\mathcal{X}_0 \oplus \sum_{\tau=t_0}^{t-1} \Phi(t,\tau+1)B(\tau)\mathcal{U}(\tau). \tag{2.4}$$

Operation '$\oplus$' is the *geometric sum*, also known as *Minkowski sum*.[2] Geometric sums and linear (or affine) transformations preserve compactness and convexity. Hence, if the initial set $\mathcal{X}_0$ and the control sets $\mathcal{U}(\tau)$, $t_0 \leq \tau < t$, are compact and convex, so is the reach set $\mathcal{X}(t,t_0,\mathcal{X}_0)$.

[2] Minkowski sum of sets $\mathcal{W}, \mathcal{Z} \subseteq \mathbf{R}^n$ is defined as $\mathcal{W} \oplus \mathcal{Z} = \{w + z \mid w \in \mathcal{W},\ z \in \mathcal{Z}\}$. $\mathcal{W} \oplus \mathcal{Z}$ is nonempty if and only if both, $\mathcal{W}$ and $\mathcal{Z}$ are nonempty. If $\mathcal{W}$ and $\mathcal{Z}$ are convex, $\mathcal{W} \oplus \mathcal{Z}$ is convex.

Definition 2.2 (Backward reach set) *The backward reach set $\mathcal{Y}(t_1, t, y_1)$ for the target position (t_1, y_1) is the set of all states $y(t)$ for which there exists some control $u(\tau, x) \in \mathcal{U}(\tau)$, $\tau = t..(t_1 - 1)$, that steers system (2.1) to the state y_1 at time step t_1. For the target set $\mathcal{Y}_1$ at time step t_1, the backward reach set $\mathcal{Y}(t_1, t, \mathcal{Y}_1)$ is*

$$\mathcal{Y}(t_1, t, \mathcal{Y}_1) = \bigcup_{y_1 \in \mathcal{Y}_1} \mathcal{Y}(t_1, t, y_1).$$

Definition 2.3 (Weak and strong invariance) *Set $\mathcal{M}$ is said to be weakly invariant with respect to the target set $\mathcal{Y}_1$ and time steps t_0 and t, if for every state $x_0 \in \mathcal{M}$ there exists a control $u(\tau, x) \in \mathcal{U}(\tau)$, $\tau = t_0..(t - 1)$, that steers the system from x_0 at time step t_0 to some state in $\mathcal{Y}_1$ at time step t.*
If all controls in $\mathcal{U}(\tau)$, $\tau = t_0..(t-1)$ steer the system from every $x_0 \in \mathcal{M}$ at time step t_0 to $\mathcal{Y}_1$ at time step t, set $\mathcal{M}$ is said to be strongly invariant with respect to $\mathcal{Y}_1$, t_0 and t.

The backward reach set $\mathcal{Y}(t_1, t, \mathcal{Y}_1)$ is the *largest weakly invariant* set with respect to the target set $\mathcal{Y}_1$ and time values t and t_1.

Remark. Backward reach set can be *uniquely* computed only if the right hand side of 2.1 is invertible, that is, there exists $f^{-1}(t, x, u)$ such that $x(t) = f^{-1}(t, x(t+1), u)$.

The following two facts about the backward reach set $\mathcal{Y}$ are similar to those for forward reach sets.

1. $\mathcal{Y}(t_1, t, \mathcal{Y}_1)$ is *the same* for open-loop and closed-loop control.
2. $\mathcal{Y}(t_1, t, \mathcal{Y}_1)$ satisfies the semigroup property,

$$\mathcal{Y}(t_1, t, \mathcal{Y}_1) = \mathcal{Y}(\tau, t, \mathcal{Y}(t_1, \tau, \mathcal{Y}_1)), \quad t \leq \tau < t_1. \tag{2.5}$$

For the linear system (2.3) the backward reach set can be expressed as

$$\mathcal{Y}(t_1, t, \mathcal{Y}_1) = \Phi(t, t_1)\mathcal{Y}_1 \oplus \sum_{\tau=t}^{t_1-1} -\Phi(t, \tau)B(\tau)\mathcal{U}(\tau). \tag{2.6}$$

This formula makes sense only for linear systems with invertible state transition matrix. Degenerate linear systems have unbounded backward reach sets that cannot be computed with available software tools.

Just as in the case of forward reach set, the backward reach set of a linear system $\mathcal{Y}(t_1, t, \mathcal{Y}_1)$ is compact and convex if the target set $\mathcal{Y}_1$ and the control sets $\mathcal{U}(\tau)$, $\tau = t..(t_1 - 1)$, are compact and convex.

Remark. In the computer science literature the reach set is said to be the result of operator *post*, and the backward reach set is the result of operator *pre*. In the control literature the backward reach set is also referred to as *solvability set*.

2.2 Systems with disturbances

Consider the discrete-time dynamical system with disturbance

$$x(t+1) = f(t, x, u, v), \tag{2.7}$$

in which we also have the disturbance input $v \in \mathbf{R}^d$ with values $v(t)$ restricted to a closed compact set $\mathcal{V}(t) \subset \mathbf{R}^d$.

In the presence of disturbances the open-loop reach set (OLRS) is different from the closed-loop reach set (CLRS).

Given the initial time step t_0, the set of initial states $\mathcal{X}_0$, and terminal time step t, there are two types of OLRS.

Definition 2.4 (OLRS of maxmin type) *The maxmin open-loop reach set* $\overline{\mathcal{X}}_{OL}(t, t_0, \mathcal{X}_0)$ *is the set of all states* x*, such that for any disturbance* $v(\tau) \in \mathcal{V}(\tau)$*, there exist an initial state* $x_0 \in \mathcal{X}_0$ *and a control* $u(\tau) \in \mathcal{U}(\tau)$*,* $t_0 \leq \tau < t$*, that steers system (2.7) from* $x(t_0) = x_0$ *to* $x(t) = x$*.*

Definition 2.5 (OLRS of minmax type) *The minmax open-loop reach set* $\underline{\mathcal{X}}_{OL}(t, t_0, \mathcal{X}_0)$ *is the set of all states* x*, such that there exists a control* $u(\tau) \in \mathcal{U}(\tau)$ *that for all disturbances* $v(\tau) \in \mathcal{V}(\tau)$*,* $t_0 \leq \tau < t$*, assigns an initial state* $x_0 \in \mathcal{X}_0$ *and steers system (2.7) from* $x(t_0) = x_0$ *to* $x(t) = x$*.*

In the maxmin case the control is chosen *after* knowing the disturbance for all the time steps $t_0..(t-1)$, whereas in the minmax case the control is chosen *before* any knowledge of the disturbance. Consequently, the OLRS does not satisfy the semigroup property.

The terms 'maxmin' and 'minmax' come from the fact that $\overline{\mathcal{X}}_{OL}(t, t_0, \mathcal{X}_0)$ is the subzero level set of the value function

$$\underline{V}(t, x) = \max_{v} \min_{u} \{ \mathbf{dist}(x(t_0), \mathcal{X}_0) \mid x(t) = x,\ u(\tau) \in \mathcal{U}(\tau),\ v(\tau) \in \mathcal{V}(\tau),\ \tau = t_0..(t-1) \}, \tag{2.8}$$

i.e., $\overline{\mathcal{X}}_{OL}(t, t_0, \mathcal{X}_0) = \{x \mid \underline{V}(t, x) \leq 0\}$, and $\underline{\mathcal{X}}_{OL}(t, t_0, \mathcal{X}_0)$ is the subzero level set of the value function

$$\overline{V}(t, x) = \min_{u} \max_{v} \{ \mathbf{dist}(x(t_0), \mathcal{X}_0) \mid x(t) = x,\ u(\tau) \in \mathcal{U}(\tau),\ v(\tau) \in \mathcal{V}(\tau),\ \tau = t_0..(t-1) \}, \tag{2.9}$$

in which $\mathbf{dist}(\cdot, \cdot)$ denotes Hausdorff semidistance.[3] Since $\underline{V}(t, x) \leq \overline{V}(t, x)$, $\underline{\mathcal{X}}_{OL}(t, t_0, \mathcal{X}_0) \subseteq \overline{\mathcal{X}}_{OL}(t, t_0, \mathcal{X}_0)$.

[3] Hausdorff semidistance between compact sets $\mathcal{W}, \mathcal{Z} \subseteq \mathbf{R}^n$ is defined as

$$\mathbf{dist}(\mathcal{W}, \mathcal{Z}) = \min\{\langle w - z, w - z \rangle^{1/2} \mid w \in \mathcal{W},\ z \in \mathcal{Z}\},$$

where $\langle \cdot, \cdot \rangle$ denotes inner product.

Note that maxmin and minmax OLRS imply *guarantees*: these are states that can be reached no matter what the disturbance is, whether it is known in advance (maxmin case) or not (minmax case). The OLRS may be empty.

Fixing some time step τ_1, $t_0 < \tau_1 < t$, define the *piecewise maxmin open-loop reach set with one correction*,

$$\overline{\mathcal{X}}^1_{OL}(t, t_0, \mathcal{X}_0) = \overline{\mathcal{X}}_{OL}(t, \tau_1, \overline{\mathcal{X}}_{OL}(\tau_1, t_0, \mathcal{X}_0)), \tag{2.10}$$

and the *piecewise minmax open-loop reach set with one correction*,

$$\underline{\mathcal{X}}^1_{OL}(t, t_0, \mathcal{X}_0) = \underline{\mathcal{X}}_{OL}(t, \tau_1, \underline{\mathcal{X}}_{OL}(\tau_1, t_0, \mathcal{X}_0)). \tag{2.11}$$

The piecewise maxmin OLRS $\overline{\mathcal{X}}^1_{OL}(t, t_0, \mathcal{X}_0)$ is the subzero level set of the value function

$$\underline{V}^1(t, x) = \max_v \min_u \{\underline{V}(\tau_1, x(\tau_1)) \mid x(t) = x,\ u(\tau) \in \mathcal{U}(\tau),\ v(\tau) \in \mathcal{V}(\tau),\ \tau = \tau_1..(t-1)\}, \tag{2.12}$$

with $V(\tau_1, x(\tau_1))$ given by (2.8), which yields

$$\underline{V}^1(t, x) \geq \underline{V}(t, x),$$

and thus,

$$\overline{\mathcal{X}}^1_{OL}(t, t_0, \mathcal{X}_0) \subseteq \overline{\mathcal{X}}_{OL}(t, t_0, \mathcal{X}_0).$$

On the other hand, the piecewise minmax OLRS $\underline{\mathcal{X}}^1_{OL}(t, t_0, \mathcal{X}_0)$ is the subzero level set of the value function

$$\overline{V}^1(t, x) = \min_u \max_v \{\overline{V}(\tau_1, x(\tau_1)) \mid x(t) = x,\ u(\tau) \in \mathcal{U}(\tau),\ v(\tau) \in \mathcal{V}(\tau),\ \tau = \tau_1..(t-1)\}, \tag{2.13}$$

with $V(\tau_1, x(\tau_1))$ given by (2.9), which yields

$$\overline{V}(t, x) \geq \overline{V}^1(t, x),$$

and thus,

$$\underline{\mathcal{X}}_{OL}(t, t_0 \mathcal{X}_0) \subseteq \underline{\mathcal{X}}^1_{OL}(t, t_0, \mathcal{X}_0).$$

We can now recurrently define piecewise maxmin and minmax OLRS with k corrections for $t_0 < \tau_1 < \cdots < \tau_k < t$. The maxmin piecewise OLRS with k corrections is

$$\overline{\mathcal{X}}^k_{OL}(t, t_0, \mathcal{X}_0) = \overline{\mathcal{X}}_{OL}(t, \tau_k, \overline{\mathcal{X}}^{k-1}_{OL}(\tau_k, t_0, \mathcal{X}_0)), \tag{2.14}$$

which is the subzero level set of the corresponding value function

$$\begin{aligned} &\underline{V}^k(t, x) = \max_v \min_u \{\underline{V}^{k-1}(\tau_k, x(\tau_k)) \mid \\ &x(t) = x,\ u(\tau) \in \mathcal{U}(\tau),\ v(\tau) \in \mathcal{V}(\tau),\ \tau = \tau_k..(t-1)\}. \end{aligned} \tag{2.15}$$

The minmax piecewise OLRS with k corrections is

$$\underline{\mathcal{X}}^k_{OL}(t, t_0, \mathcal{X}_0) = \underline{\mathcal{X}}_{OL}(t, \tau_k, \underline{\mathcal{X}}^{k-1}_{OL}(\tau_k, t_0, \mathcal{X}_0)), \tag{2.16}$$

which is the subzero level set of the corresponding value function

$$\begin{aligned}&\overline{V}^k(t,x) = \min_u \max_v \{\overline{V}^{k-1}(\tau_k, x(\tau_k)) \mid \\ &x(t) = x,\ u(\tau) \in \mathcal{U}(\tau),\ v(\tau) \in \mathcal{V}(\tau),\ \tau = \tau_k..(t-1)\}.\end{aligned} \tag{2.17}$$

From (2.12), (2.13), (2.15) and (2.17) it follows that

$$\underline{V}(t,x) \leq \underline{V}^1(t,x) \leq \cdots \leq \underline{V}^k(t,x) \leq \overline{V}^k(t,x) \leq \cdots \leq \overline{V}^1(t,x) \leq \overline{V}(t,x).$$

Hence,

$$\begin{aligned}&\underline{\mathcal{X}}_{OL}(t,t_0,\mathcal{X}_0) \subseteq \underline{\mathcal{X}}^1_{OL}(t,t_0,\mathcal{X}_0) \subseteq \cdots \subseteq \underline{\mathcal{X}}^k_{OL}(t,t_0,\mathcal{X}_0) \subseteq \\ &\overline{\mathcal{X}}^k_{OL}(t,t_0,\mathcal{X}_0) \subseteq \cdots \subseteq \overline{\mathcal{X}}^1_{OL}(t,t_0,\mathcal{X}_0) \subseteq \overline{\mathcal{X}}_{OL}(t,t_0,\mathcal{X}_0).\end{aligned} \tag{2.18}$$

We call

$$\overline{\mathcal{X}}_{CL}(t,t_0,\mathcal{X}_0) = \overline{\mathcal{X}}^{t-t_0-1}_{OL}(t,t_0,\mathcal{X}_0) \tag{2.19}$$

the *maxmin closed-loop reach set* of system (2.7) at time step t, and we call

$$\underline{\mathcal{X}}_{CL}(t,t_0,\mathcal{X}_0) = \underline{\mathcal{X}}^{t-t_0-1}_{OL}(t,t_0,\mathcal{X}_0) \tag{2.20}$$

the *minmax closed-loop reach set* of system (2.7) at time step t.

Definition 2.6 (CLRS of maxmin type) *Given initial time step t_0 and the set of initial states $\mathcal{X}_0$, the maxmin CLRS $\overline{\mathcal{X}}_{CL}(t,t_0,\mathcal{X}_0)$ of system (2.7) at time step $t > t_0$, is the set of all states x, for each of which and for every disturbance $v(\tau) \in \mathcal{V}(\tau)$, there exist an initial state $x_0 \in \mathcal{X}_0$ and a control $u(\tau, x(\tau)) \in \mathcal{U}(\tau)$, such that the trajectory $x(\tau|v(\tau), u(\tau, x(\tau)))$ satisfying $x(t_0) = x_0$ and*

$$x(\tau+1|v(\tau), u(\tau,x(\tau))) \in f(\tau, x(\tau), u(\tau,x(\tau)), v(\tau)),$$

with $\tau = t_0..(t-1)$, is such that $x(t) = x$.

Definition 2.7 (CLRS of minmax type) *Given initial time step t_0 and the set of initial states $\mathcal{X}_0$, the maxmin CLRS $\underline{\mathcal{X}}_{CL}(t,t_0,\mathcal{X}_0)$ of system (2.7), at time step $t > t_0$, is the set of all states x, for each of which there exists a control $u(\tau, x(\tau)) \in \mathcal{U}(\tau)$, such that for every disturbance $v(\tau) \in \mathcal{V}(\tau)$ there exists an initial state $x_0 \in \mathcal{X}_0$ and trajectory $x(\tau, v(\tau)|u(\tau, x(\tau)))$ satisfying*

$$x(\tau+1, v(\tau)|u(\tau,x(\tau))) \in f(\tau, x(\tau), u(\tau,x(\tau)), v(\tau)),$$

with $\tau = t_0..(t-1)$, $x(t_0) = x_0$ and $x(t) = x$.

By construction, both maxmin and minmax CLRS satisfy the semigroup property (2.2).

Remark. In case the initial set $\mathcal{X}_0 = \{x_0\}$ (is limited to one state) and the disturbance is unknown ($\mathcal{V}(\tau)$ contains more than one point for some $t_0 \le \tau < t$), the minmax OLRS is empty. To fix the situation, one has to work with a neighborhood of x_0 as initial set.

Consider the linear system case,

$$f(t, x, u) = A(t)x(t) + B(t)u + G(t)v, \tag{2.21}$$

where $A(t)$ and $B(t)$ are as in (2.3), and $G(t)$ takes its values in $\mathbf{R}^d$.

The maxmin OLRS can be expressed through set-valued sums,

$$\begin{aligned}&\overline{\mathcal{X}}_{OL}(t, t_0, \mathcal{X}_0) = \\ &\left(\Phi(t, t_0)\mathcal{X}_0 \oplus \textstyle\sum_{\tau=t_0}^{t-1} \Phi(t, \tau+1)B(\tau)\mathcal{U}(\tau)\right) \dot{-} \\ &\textstyle\sum_{\tau=t_0}^{t-1} \Phi(t, \tau+1)(-G(\tau))\mathcal{V}(\tau).\end{aligned} \tag{2.22}$$

Similarly, the minmax OLRS is

$$\begin{aligned}&\underline{\mathcal{X}}_{OL}(t, t_0, \mathcal{X}_0) = \\ &\left(\Phi(t, t_0)\mathcal{X}_0 \dot{-} \textstyle\sum_{\tau=t_0}^{t-1} \Phi(t, \tau+1)(-G(\tau))\mathcal{V}(\tau)\right) \oplus \\ &\textstyle\sum_{\tau=t_0}^{t-1} \Phi(t, \tau+1)B(\tau)\mathcal{U}(\tau).\end{aligned} \tag{2.23}$$

The operation '$\dot{-}$' is *geometric difference*, also known as *Minkowski difference*.[4]

Now consider the piecewise OLRS with k corrections. Expression (2.14) translates into

$$\begin{aligned}&\overline{\mathcal{X}}_{OL}^k(t, t_0, \mathcal{X}_0) = \\ &\left(\Phi(t, \tau_k)\overline{\mathcal{X}}_{OL}^{k-1}(\tau_k, t_0, \mathcal{X}_0) \oplus \textstyle\sum_{\tau=\tau_k}^{t-1} \Phi(t, \tau+1)B(\tau)\mathcal{U}(\tau)\right) \dot{-} \\ &\textstyle\sum_{\tau=\tau_k}^{t-1} \Phi(t, \tau+1)(-G(\tau))\mathcal{V}(\tau).\end{aligned} \tag{2.24}$$

Expression (2.16) translates into

$$\begin{aligned}&\underline{\mathcal{X}}_{OL}^k(t, t_0, \mathcal{X}_0) = \\ &\left(\Phi(t, \tau_k)\underline{\mathcal{X}}_{OL}^{k-1}(\tau_k, t_0, \mathcal{X}_0) \dot{-} \textstyle\sum_{\tau=\tau_k}^{t-1} \Phi(t, \tau+1)(-G(\tau))\mathcal{V}(\tau)\right) \oplus \\ &\textstyle\sum_{\tau=\tau_k}^{t-1} \Phi(t, \tau+1)B(\tau)\mathcal{U}(\tau).\end{aligned} \tag{2.25}$$

Since for any $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3 \subseteq \mathbf{R}^n$ it is true that

$$(\mathcal{W}_1 \dot{-} \mathcal{W}_2) \oplus \mathcal{W}_3 = (\mathcal{W}_1 \oplus \mathcal{W}_3) \dot{-} (\mathcal{W}_2 \oplus \mathcal{W}_3) \subseteq (\mathcal{W}_1 \oplus \mathcal{W}_3) \dot{-} \mathcal{W}_2,$$

from (2.24), (2.25), it is clear that (2.18) is true.

For linear systems, if the initial set $\mathcal{X}_0$, control bounds $\mathcal{U}(\tau)$ and disturbance bounds $\mathcal{V}(\tau)$, $\tau = t_0..(t-1)$, are compact and convex, the CLRS $\overline{\mathcal{X}}_{CL}(t, t_0, \mathcal{X}_0)$ and $\underline{\mathcal{X}}_{CL}(t, t_0, \mathcal{X}_0)$ are compact and convex, provided they are nonempty.

Just as for forward reach sets, the backward reach sets can be open-loop (OLBRS) or closed-loop (CLBRS).

[4] The Minkowski difference of sets $\mathcal{W}, \mathcal{Z} \in \mathbf{R}^n$ is defined as $\mathcal{W} \dot{-} \mathcal{Z} = \{\xi \in \mathbf{R}^n \mid \xi \oplus \mathcal{Z} \subseteq \mathcal{W}\}$. If $\mathcal{W}$ and $\mathcal{Z}$ are convex, $\mathcal{W} \dot{-} \mathcal{Z}$ is convex if it is nonempty.

Definition 2.8 (OLBRS of maxmin type) *Given the terminal time step t_1 and target set $\mathcal{Y}_1$, the maxmin open-loop backward reach set $\overline{\mathcal{Y}}_{OL}(t_1, t, \mathcal{Y}_1)$ of system (2.7) at time step $t < t_1$, is the set of all y, such that for any disturbance $v(\tau) \in \mathcal{V}(\tau)$ there exists a terminal state $y_1 \in \mathcal{Y}_1$ and control $u(\tau) \in \mathcal{U}(\tau)$, $\tau = t..(t_1 - 1)$, which steers the system from $y(t) = y$ to $y(t_1) = y_1$.*

$\overline{\mathcal{Y}}_{OL}(t_1, t, \mathcal{Y}_1)$ is the subzero level set of the value function

$$\begin{aligned}&\underline{V}_b(t, y) = \max_v \min_u \{\mathbf{dist}(y(t_1), \mathcal{Y}_1) \mid \\ &y(t) = y,\ u(\tau) \in \mathcal{U}(\tau),\ v(\tau) \in \mathcal{V}(\tau),\ \tau = t..(t_1 - 1)\},\end{aligned} \tag{2.26}$$

Definition 2.9 (OLBRS of minmax type) *Given the terminal time step t_1 and target set $\mathcal{Y}_1$, the minmax open-loop backward reach set $\underline{\mathcal{Y}}_{OL}(t_1, t, \mathcal{Y}_1)$ of system (2.7) at time step $t < t_1$, is the set of all y, such that there exists a control $u(\tau) \in \mathcal{U}(\tau)$ that for all disturbances $v(\tau \in \mathcal{V}(\tau)$, $\tau = t..(t_1 - 1)$, assigns a terminal state $y_1 \in \mathcal{Y}_1$ and steers the system from $y(t) = y$ to $y(t_1) = y_1$.*

$\underline{\mathcal{Y}}_{OL}(t_1, t, \mathcal{Y}_1)$ is the subzero level set of the value function

$$\begin{aligned}&\overline{V}_b(t, y) = \min_u \max_v \{\mathbf{dist}(y(t_1), \mathcal{Y}_1) \mid \\ &y(t) = y,\ u(\tau) \in \mathcal{U}(\tau),\ v(\tau) \in \mathcal{V}(\tau),\ \tau = t..(t_1 - 1)\},\end{aligned} \tag{2.27}$$

Both, maxmin and minmax OLBRS are *weakly invariant* with respect to the target set and the control. Maxmin OLBRS is also *weakly invariant* with respect to the target set and the disturbance, whereas minmax OLBRS is *strongly invariant* with respect to the target set and the disturbance.

Remark. The backward reach set can be computed only if the right hand side of (2.7) is invertible.

Remark. As in the forward reachability case, if the target set is comprised of a single state and the disturbance is unknown, the minmax OLBRS is empty, and one has to consider some neighborhood of the target state as a target set.

Similarly to the forward reachability case, we construct piecewise OLBRS with one correction at time step τ_1, $t < \tau_1 < t_1$. The piecewise maxmin OLBRS with one correction is

$$\overline{\mathcal{Y}}^1_{OL}(t_1, t, \mathcal{Y}_1) = \overline{\mathcal{Y}}_{OL}(\tau_1, t, \overline{\mathcal{Y}}_{OL}(t_1, \tau_1, \mathcal{Y}_1)), \tag{2.28}$$

and it is the subzero level set of the function

$$\begin{aligned}&\underline{V}^1_b(t, y) = \max_v \min_u \{\underline{V}_b(\tau_1, y(\tau_1)) \mid \\ &y(t) = y,\ u(\tau) \in \mathcal{U}(\tau),\ v(\tau) \in \mathcal{V}(\tau),\ \tau = t..(\tau_1 - 1)\}.\end{aligned} \tag{2.29}$$

The piecewise minmax OLBRS with one correction is

$$\underline{\mathcal{Y}}_{OL}^1(t_1,t,\mathcal{Y}_1) = \underline{\mathcal{Y}}_{OL}(\tau_1,t,\underline{\mathcal{Y}}_{OL}(t_1,\tau_1,\mathcal{Y}_1)), \tag{2.30}$$

and it is the subzero level set of the function

$$\begin{aligned}&\overline{V}_b^1(t,y) = \min_u \max_v \{\overline{V}_b(\tau_1, y(\tau_1)) \mid \\ &y(t)=y,\ u(\tau)\in\mathcal{U}(\tau),\ v(\tau)\in\mathcal{V}(\tau),\ \tau = t..(\tau_1-1)\},\end{aligned} \tag{2.31}$$

Recursively define maxmin and minmax OLBRS with k corrections for $t < \tau_k < \cdots < \tau_1 < t_1$. The maxmin OLBRS with k corrections is

$$\overline{\mathcal{Y}}_{OL}^k(t_1,t,\mathcal{Y}_1) = \overline{\mathcal{Y}}_{OL}(\tau_k,t,\overline{\mathcal{Y}}_{OL}^{k-1}(t_1,\tau_k,\mathcal{Y}_1)), \tag{2.32}$$

which is the subzero level set of function

$$\begin{aligned}&\underline{V}_b^k(t,y) = \max_v \min_u \{\underline{V}_b^{k-1}(\tau_k, y(\tau_k)) \mid \\ &y(t)=y,\ u(\tau)\in\mathcal{U}(\tau),\ v(\tau)\in\mathcal{V}(\tau),\ \tau = t..(\tau_k-1)\}.\end{aligned} \tag{2.33}$$

The minmax OLBRS with k corrections is

$$\underline{\mathcal{Y}}_{OL}^k(t_1,t,\mathcal{Y}_1) = \underline{\mathcal{Y}}_{OL}(\tau_k,t,\underline{\mathcal{Y}}_{OL}^{k-1}(t_1,\tau_k,\mathcal{Y}_1)), \tag{2.34}$$

which is the subzero level set of the function

$$\begin{aligned}&\overline{V}_b^k(t,y) = \min_u \max_v \{\overline{V}_b^{k-1}(\tau_k, y(\tau_k)) \mid \\ &y(t)=y,\ u(\tau)\in\mathcal{U}(\tau),\ v(\tau)\in\mathcal{V}(\tau),\ \tau = t..(\tau_k-1)\},\end{aligned} \tag{2.35}$$

From (2.29), (2.31), (2.33) and (2.35) it follows that

$$\underline{V}_b(t,y) \le \underline{V}_b^1(t,y) \le \cdots \le \underline{V}_b^k(t,y) \le \overline{V}_b^k(t,y) \le \cdots \le \overline{V}_b^1(t,y) \le \overline{V}_b(t,y).$$

Hence,

$$\begin{aligned}&\underline{\mathcal{Y}}_{OL}(t_1,t,\mathcal{Y}_1) \subseteq \underline{\mathcal{Y}}_{OL}^1(t_1,t,\mathcal{Y}_1) \subseteq \cdots \subseteq \underline{\mathcal{Y}}_{OL}^k(t_1,t,\mathcal{Y}_1) \subseteq \\ &\overline{\mathcal{Y}}_{OL}^k(t_1,t,\mathcal{Y}_1) \subseteq \cdots \subseteq \overline{\mathcal{Y}}_{OL}^1(t_1,t,\mathcal{Y}_1) \subseteq \overline{\mathcal{Y}}_{OL}(t_1,t,\mathcal{Y}_1).\end{aligned} \tag{2.36}$$

We say that

$$\overline{\mathcal{Y}}_{CL}(t_1,t,\mathcal{Y}_1) = \overline{\mathcal{Y}}_{OL}^{t_1-t-1}(t_1,t,\mathcal{Y}_1) \tag{2.37}$$

is the *maxmin closed-loop backward reach set* of system (2.7) at time step t.

We say that

$$\underline{\mathcal{Y}}_{CL}(t_1,t,\mathcal{Y}_1) = \underline{\mathcal{Y}}_{OL}^{t_1-t-1}(t_1,t,\mathcal{Y}_1) \tag{2.38}$$

is the *minmax closed-loop backward reach set* of system (2.7) at time step t.

Definition 2.10 (CLBRS of maxmin type) *Given the terminal time step t_1 and target set $\mathcal{Y}_1$, the maxmin CLBRS $\overline{\mathcal{Y}}_{CL}(t_1, t, \mathcal{Y}_1)$ of system (2.7) at time step $t < t_1$, is the set of all states y, for each of which for every disturbance $v(\tau) \in \mathcal{V}(\tau)$ there exists terminal state $y_1 \in \mathcal{Y}_1$ and control $u(\tau, y(\tau)) \in \mathcal{U}(\tau)$ that assigns trajectory $y(\tau, |v(\tau), u(\tau, y(\tau)))$ satisfying*

$$y(\tau+1|v(\tau), u(\tau, y(\tau))) \in f(\tau, y(\tau), u(\tau, y(\tau)), v(\tau)),$$

with $\tau = t..(t_1 - 1)$, such that $y(t) = y$ and $y(t_1) = y_1$.

Definition 2.11 (CLBRS of minmax type) *Given the terminal time step t_1 and target set $\mathcal{Y}_1$, the minmax CLBRS $\underline{\mathcal{Y}}_{CL}(t_1, t, \mathcal{Y}_1)$ of system (2.7) at time step $t < t_1$, is the set of all states y, for each of which there exists control $u(\tau, y(\tau)) \in \mathcal{U}(\tau)$ that for every disturbance $v(\tau) \in \mathcal{V}(\tau)$ assigns terminal state $y_1 \in \mathcal{Y}_1$ and trajectory $y(\tau, v(\tau)|u(\tau, y(\tau)))$ satisfying*

$$y(\tau+1, v(\tau)|u(\tau, y(\tau))) \in f(\tau, y(\tau), u(\tau, y(\tau)), v(\tau)),$$

with $\tau = t..(t_1 - 1)$, such that $y(t) = y$ and $y(t_1) = y_1$.

Both maxmin and minmax CLBRS satisfy the semigroup property (2.5).

For linear systems, the maxmin OLBRS can be expressed through set-valued sums,

$$\begin{aligned}&\overline{\mathcal{Y}}_{OL}(t_1, t, \mathcal{Y}_1) = \\ &\left(\Phi(t, t_1)\mathcal{Y}_1 \oplus \textstyle\sum_{\tau=t}^{t_1-1} -\Phi(t, \tau+1)B(\tau)\mathcal{U}(\tau)\right) \dot{-} \\ &\textstyle\sum_{\tau=t}^{t_1-1} \Phi(t, \tau+1)G(\tau)\mathcal{V}(\tau).\end{aligned} \tag{2.39}$$

Similarly, the minmax OLBRS is

$$\begin{aligned}&\underline{\mathcal{Y}}_{OL}(t_1, t, \mathcal{Y}_1) = \\ &\left(\Phi(t, t_1)\mathcal{Y}_1 \dot{-} \textstyle\sum_{\tau=t}^{t_1-1} \Phi(t, \tau+1)G(\tau)\mathcal{V}(\tau)\right) \oplus \\ &\textstyle\sum_{\tau=t}^{t_1-1} -\Phi(t, \tau+1)B(\tau)\mathcal{U}(\tau).\end{aligned} \tag{2.40}$$

Now consider piecewise OLBRS with k corrections. Expression (2.32) translates into

$$\begin{aligned}&\overline{\mathcal{Y}}^k_{OL}(t_1, t, \mathcal{Y}_1) = \\ &\left(\Phi(t, \tau_k)\overline{\mathcal{Y}}^{k-1}_{OL}(t_1, \tau_k, \mathcal{Y}_1) \oplus \textstyle\sum_{\tau=t}^{\tau_k-1} -\Phi(t, \tau+1)B(\tau)\mathcal{U}(\tau)\right) \dot{-} \\ &\textstyle\sum_{\tau=t}^{\tau_k-1} \Phi(t, \tau+1)G(\tau)\mathcal{V}(\tau).\end{aligned} \tag{2.41}$$

Expression (2.34) translates into

$$\begin{aligned}&\underline{\mathcal{Y}}^k_{OL}(t_1, t, \mathcal{Y}_1) = \\ &\left(\Phi(t, \tau_k)\overline{\mathcal{Y}}^{k-1}_{OL}(t_1, \tau_k, \mathcal{Y}_1) \dot{-} \textstyle\sum_{\tau=t}^{\tau_k-1} \Phi(t, \tau+1)G(\tau)\mathcal{V}(\tau)\right) \oplus \\ &\textstyle\sum_{\tau=t}^{\tau_k-1} -\Phi(t, \tau+1)B(\tau)\mathcal{U}(\tau).\end{aligned} \tag{2.42}$$

Computation of backward reach sets for discrete-time linear systems makes sense only if the state transition matrix $\Phi(t_1, t)$ is invertible.

If the target set $\mathcal{Y}_1$, control sets $\mathcal{U}(\tau)$ and disturbance sets $\mathcal{V}(\tau)$, $\tau = t..(t_1 - 1)$, are compact and convex, then CLBRS $\overline{\mathcal{Y}}_{CL}(t_1, t, \mathcal{Y}_1)$ and $\underline{\mathcal{Y}}_{CL}(t_1, t, \mathcal{Y}_1)$ are compact and convex, if they are nonempty.

2.3 Reachability problem

Reachability analysis is concerned with the computation of the forward $\mathcal{X}(t, t_0, \mathcal{X}_0)$ and backward $\mathcal{Y}(t_1, t, \mathcal{Y}_1)$ reach sets (the reach sets may be maxmin or minmax) in a way that can effectively meet requests like the following:

1. For the given number of time steps $t_0..t$, determine whether the system can be steered into the given target set $\mathcal{Y}_1$. In other words, is the set $\mathcal{Y}_1 \cap \bigcup_{t_0 \leq \tau \leq t} \mathcal{X}(\tau, t_0, \mathcal{X}_0)$ nonempty? And if the answer is 'yes', find a control that steers the system to the target set (or avoids the target set).

   So-called verification problems often consist in ensuring that the system is unable to reach an "unsafe" target set within a given time interval.

2. If the target set $\mathcal{Y}_1$ is reachable from the given initial condition $\{t_0, \mathcal{X}_0\}$ in time steps $t_0..t$, find the shortest time to reach $\mathcal{Y}_1$,

$$\arg\min_{\tau}\{\mathcal{X}(\tau, t_0, \mathcal{X}_0) \cap \mathcal{Y}_1 \neq \emptyset \mid t_0 \leq \tau \leq t\}.$$

3. Given the terminal time step t_1, target set $\mathcal{Y}_1$ and time step $t < t_1$ find the set of states starting at time step t from which the system can reach $\mathcal{Y}_1$ within time steps $t..t_1$. In other words, find $\bigcup_{t \leq \tau < t_1} \mathcal{Y}(t_1, \tau, \mathcal{Y}_1)$.

4. Find a closed-loop control that steers a system with disturbances to the given target set in given time.

5. Graphically display the projection of the reach set along any specified two- or three-dimensional subspace.

For linear systems, if the initial set $\mathcal{X}_0$, target set $\mathcal{Y}_1$, control bounds $\mathcal{U}(\cdot)$ and disturbance bounds $\mathcal{V}(\cdot)$ are compact and convex, so are the forward $\mathcal{X}(t, t_0, \mathcal{X}_0)$ and backward $\mathcal{Y}(t_1, t, \mathcal{Y}_1)$ reach sets. Hence reachability analysis requires the computationally effective manipulation of convex sets, and performing the set-valued operations of unions, intersections, geometric sums and differences.

Existing reach set computation tools can deal reliably only with linear systems with convex constraints. A claim that certain tool or method can be used *effectively* for nonlinear systems must be treated with caution, and the first question to ask is for what class of nonlinear systems and with what limit on the state space dimension does this tool work? Some "reachability methods for nonlinear systems" reduce to the local linearization of a system followed by the use of well-tested techniques for linear system reach set computation. Thus these approaches in fact use reachability methods for linear systems. We refer the reader to the Ellipsoidal Toolbox manual [3] for the review of existing reachability methods and tools.

3 Facts from ellipsoidal calculus

3.1 Basic definitions

We start with basic definitions.

Definition 3.1 (Ellipsoid) *Ellipsoid $\mathcal{E}(q,Q)$ in $\mathbf{R}^n$ with center q and shape matrix Q is the set*

$$\mathcal{E}(q,Q) = \{x \in \mathbf{R}^n \mid \langle (x-q), Q^{-1}(x-q)\rangle \leq 1\}, \tag{3.1}$$

wherein Q is positive definite ($Q = Q^T$ and $\langle x, Qx\rangle > 0$ for all nonzero $x \in \mathbf{R}^n$).

Here $\langle \cdot, \cdot \rangle$ denotes inner product.

Definition 3.2 (Support function) *The support function of a set $\mathcal{X} \subseteq \mathbf{R}^n$ is*

$$\rho(l \mid \mathcal{X}) = \sup_{x\in\mathcal{X}} \langle l, x\rangle.$$

In particular, the support function of the ellipsoid (3.1) is

$$\rho(l \mid \mathcal{E}(q,Q)) = \langle l, q\rangle + \langle l, Ql\rangle^{1/2}. \tag{3.2}$$

Although in (3.1) Q is assumed to be positive definite, in practice we often deal with situations when Q is singular, that is, with degenerate ellipsoids flat in those directions for which the corresponding eigenvalues are zero. Therefore, it is useful to give an alternative definition of an ellipsoid using the expression (3.2).

Definition 3.3 (Ellipsoid defined through support function) *Ellipsoid $\mathcal{E}(q,Q)$ in $\mathbf{R}^n$ with center q and shape matrix Q is the set*

$$\mathcal{E}(q,Q) = \{x \in \mathbf{R}^n \mid \langle l, x\rangle \leq \langle l, q\rangle + \langle l, Ql\rangle^{1/2} \text{ for all } l \in \mathbf{R}^n\}, \tag{3.3}$$

wherein matrix Q is positive semidefinite ($Q = Q^T$ and $\langle x, Qx\rangle \geq 0$ for all $x \in \mathbf{R}^n$).

3.2 Affine transformation

The simplest operation with ellipsoids is an affine transformation. Let ellipsoid $\mathcal{E}(q,Q) \subseteq \mathbf{R}^n$, matrix $A \in \mathbf{R}^{m\times n}$ and vector $b \in \mathbf{R}^m$. Then

$$A\mathcal{E}(q,Q) + b = \mathcal{E}(Aq + b, AQA^T). \tag{3.4}$$

Thus, ellipsoids are preserved under affine transformation. If the rows of A are linearly independent (which implies $m \leq n$), and $b = 0$, the affine transformation is called *projection*.

3.3 Geometric sum

Consider the geometric sum $\mathcal{E}(q_1, Q_1) \oplus \cdots \oplus \mathcal{E}(q_k, Q_k) \subseteq \mathbf{R}^n$. The resulting set is not generally an ellipsoid. However, it can be tightly approximated by the parametrized families of external and internal ellipsoids.

Let parameter l be some nonzero vector in $\mathbf{R}^n$. Then the external approximation $\mathcal{E}(q, Q_l^+)$ and the internal approximation $\mathcal{E}(q, Q_l^-)$ of the sum $\mathcal{E}(q_1, Q_1) \oplus \cdots \oplus \mathcal{E}(q_k, Q_k)$ are *tight* along direction l, i.e.,

$$\mathcal{E}(q, Q_l^-) \subseteq \mathcal{E}(q_1, Q_1) \oplus \cdots \oplus \mathcal{E}(q_k, Q_k) \subseteq \mathcal{E}(q, Q_l^+)$$

and the support function

$$\rho(\pm l \mid \mathcal{E}(q, Q_l^-)) = \rho(\pm l \mid \mathcal{E}(q_1, Q_1) \oplus \cdots \oplus \mathcal{E}(q_k, Q_k)) = \rho(\pm l \mid \mathcal{E}(q, Q_l^+)).$$

Here the center q is

$$q = q_1 + \cdots + q_k, \tag{3.5}$$

the shape matrix of the external ellipsoid Q_l^+ is

$$Q_l^+ = \left(\langle l, Q_1 l\rangle^{1/2} + \cdots + \langle l, Q_k l\rangle^{1/2}\right)\left(\frac{1}{\langle l, Q_1 l\rangle^{1/2}} Q_1 + \cdots + \frac{1}{\langle l, Q_k l\rangle^{1/2}} Q_k\right), \tag{3.6}$$

and the shape matrix of the internal ellipsoid Q_l^- is

$$Q_l^- = \left(Q_1^{1/2} + S_2 Q_2^{1/2} + \cdots + S_k Q_k^{1/2}\right)^T \left(Q_1^{1/2} + S_2 Q_2^{1/2} + \cdots + S_k Q_k^{1/2}\right), \tag{3.7}$$

with matrices S_i, $i = 2, \cdots, k$, being orthogonal ($S_i S_i^T = I$) and such that vectors $Q_1^{1/2} l, S_2 Q_2^{1/2} l, \cdots, S_k Q_k^{1/2} l$ are collinear.

Varying vector l we get exact external and internal approximations,

$$\bigcup_{\langle l, l\rangle = 1} \mathcal{E}(q, Q_l^-) = \mathcal{E}(q_1, Q_1) \oplus \cdots \oplus \mathcal{E}(q_k, Q_k) = \bigcap_{\langle l, l\rangle = 1} \mathcal{E}(q, Q_l^+).$$

For proofs of formulas given in this section, see [20, 22].

One last comment is about how to find orthogonal matrices $S_2, \cdots, S_k$ that align vectors $Q_2^{1/2} l, \cdots, Q_k^{1/2} l$ with $Q_1^{1/2} l$. Let v and w be some unit vectors in $\mathbf{R}^n$. We have to find matrix S such that $Sv = w$. For that, we perform singular value decomposition (SVD) of vectors v and w:

$$v = U_v \Sigma_v W_v^T, \quad w = U_w \Sigma_w W_w^T. \tag{3.8}$$

Notice that W_v and W_w are ± 1 scalars. The matrix S is now easily determined:

$$S U_v W_v = U_w W_w \quad \Rightarrow \quad S = U_w W_w W_v U_v^T. \tag{3.9}$$

3.4 Geometric difference

Consider the geometric difference of nondegenerate ellipsoids $\mathcal{E}(q_1, Q_1)\dot{-}\mathcal{E}(q_2, Q_2) \subseteq \mathbf{R}^n$. We say that ellipsoid $\mathcal{E}(q_1, Q_1)$ is *bigger* than ellipsoid $\mathcal{E}(q_2, Q_2)$ if

$$\mathcal{E}(0, Q_2) \subseteq \mathcal{E}(0, Q_1).$$

If this condition is not fulfilled, the geometric difference $\mathcal{E}(q_1, Q_1)\dot{-}\mathcal{E}(q_2, Q_2)$ is an empty set:

$$\mathcal{E}(0, Q_2) \not\subseteq \mathcal{E}(0, Q_1) \quad \Rightarrow \quad \mathcal{E}(q_1, Q_1)\dot{-}\mathcal{E}(q_2, Q_2) = \emptyset.$$

If $\mathcal{E}(q_1, Q_1)$ is bigger than $\mathcal{E}(q_2, Q_2)$ and $\mathcal{E}(q_2, Q_2)$ is bigger than $\mathcal{E}(q_1, Q_1)$, in other words, if $Q_1 = Q_2$,

$$\mathcal{E}(q_1, Q_1)\dot{-}\mathcal{E}(q_2, Q_2) = \{q_1 - q_2\} \quad \text{and} \quad \mathcal{E}(q_2, Q_2)\dot{-}\mathcal{E}(q_1, Q_1) = \{q_2 - q_1\}.$$

To check if ellipsoid $\mathcal{E}(q_1, Q_1)$ is bigger than ellipsoid $\mathcal{E}(q_2, Q_2)$, we perform simultaneous diagonalization of matrices Q_1 and Q_2, that is, we find matrix T such that

$$TQ_1T^T = I \quad \text{and} \quad TQ_2T^T = D,$$

where D is some diagonal matrix. Simultaneous diagonalization of Q_1 and Q_2 is possible because both are symmetric positive definite (see [14]). To find such matrix T, we first do the SVD of Q_1:

$$Q_1 = U_1\Sigma_1W_1^T. \tag{3.10}$$

Then the SVD of matrix $\Sigma_1^{-1/2}U_1^TQ_2U_1\Sigma_1^{-1/2}$:

$$\Sigma_1^{-1/2}U_1^TQ_2U_1\Sigma_1^{-1/2} = U_2\Sigma_2W_2^T. \tag{3.11}$$

Now, T is defined as

$$T = U_2^T\Sigma_1^{-1/2}U_1^T. \tag{3.12}$$

If the biggest diagonal element (eigenvalue) of matrix $D = TQ_2T^T$ is less than or equal to 1, $\mathcal{E}(0, Q_2) \subseteq \mathcal{E}(0, Q_1)$.

Once it is established that ellipsoid $\mathcal{E}(q_1, Q_1)$ is bigger than ellipsoid $\mathcal{E}(q_2, Q_2)$, we know that the geometric difference $\mathcal{E}(q_1, Q_1)\dot{-}\mathcal{E}(q_2, Q_2)$ is a nonempty convex compact set. Although it is not generally an ellipsoid, we can find tight external and internal approximations of this set parametrized by vector $l \in \mathbf{R}^n$. Unlike geometric sum, however, ellipsoidal approximations for the geometric difference do not exist for every direction l. Vectors for which the approximations do not exist are called *bad directions*.

Given two ellipsoids $\mathcal{E}(q_1, Q_1)$ and $\mathcal{E}(q_2, Q_2)$ with $\mathcal{E}(0, Q_2) \subseteq \mathcal{E}(0, Q_1)$, l is a bad direction if

$$\frac{\langle l, Q_1l\rangle^{1/2}}{\langle l, Q_2l\rangle^{1/2}} > r,$$

in which r is a minimal root of the equation

$$\mathbf{det}(Q_1 - rQ_2) = 0.$$

To find r, compute matrix T by (3.10-3.12) and define

$$r = \frac{1}{\max(\mathbf{diag}(TQ_2T^T))}.$$

If l is *not* a bad direction, we can find tight external and internal ellipsoidal approximations $\mathcal{E}(q, Q_l^+)$ and $\mathcal{E}(q, Q_l^-)$ such that

$$\mathcal{E}(q, Q_l^-) \subseteq \mathcal{E}(q_1, Q_1)\dot{-}\mathcal{E}(q_2, Q_2) \subseteq \mathcal{E}(q, Q_l^+)$$

and

$$\rho(\pm l \mid \mathcal{E}(q, Q_l^-)) = \rho(\pm l \mid \mathcal{E}(q_1, Q_1)\dot{-}\mathcal{E}(q_2, Q_2)) = \rho(\pm l \mid \mathcal{E}(q, Q_l^+)).$$

The center q is

$$q = q_1 - q_2; \tag{3.13}$$

the shape matrix of the internal ellipsoid Q_l^- is

$$Q_l^- = \left(1 - \frac{\langle l, Q_1 l\rangle^{1/2}}{\langle l, Q_2 l\rangle^{1/2}}\right) Q_1 + \left(1 - \frac{\langle l, Q_2 l\rangle^{1/2}}{\langle l, Q_1 l\rangle^{1/2}}\right) Q_2; \tag{3.14}$$

and the shape matrix of the external ellipsoid Q_l^+ is

$$Q_l^+ = \left(Q_1^{1/2} + SQ_2^{1/2}\right)^T \left(Q_1^{1/2} + SQ_2^{1/2}\right). \tag{3.15}$$

Here S is an orthogonal matrix such that vectors $Q_1^{1/2}l$ and $SQ_2^{1/2}l$ are collinear. S is found from (3.8-3.9), with $v = Q_2^{1/2}l$ and $w = Q_1^{1/2}l$.

Running l over all unit vectors that are not bad directions, we get

$$\bigcup_{\langle l,l\rangle=1} \mathcal{E}(q, Q_l^-) = \mathcal{E}(q_1, Q_1)\dot{-}\mathcal{E}(q_2, Q_2) = \bigcap_{\langle l,l\rangle=1} \mathcal{E}(q, Q_l^+).$$

For proofs of formulas given in this section, see [20].

3.5 Geometric difference-sum

Given ellipsoids $\mathcal{E}(q_1, Q_1)$, $\mathcal{E}(q_2, Q_2)$ and $\mathcal{E}(q_3, Q_3)$, it is possible to compute families of external and internal approximating ellipsoids for

$$\mathcal{E}(q_1, Q_1)\dot{-}\mathcal{E}(q_2, Q_2) \oplus \mathcal{E}(q_3, Q_3) \tag{3.16}$$

parametrized by direction l, if this set is nonempty ($\mathcal{E}(0, Q_2) \subseteq \mathcal{E}(0, Q_1)$).

First, using the result of the previous section, for any direction l that is not bad, we obtain tight external $\mathcal{E}(q_1 - q_2, Q_l^{0+})$ and internal $\mathcal{E}(q_1 - q_2, Q_l^{0-})$ approximations of the set $\mathcal{E}(q_1, Q_1)\dot{-}\mathcal{E}(q_2, Q_2)$.

The second and last step is, using the result of Section 3.3, to find tight external ellipsoidal approximation $\mathcal{E}(q_1 - q_2 + q_3, Q_l^{+})$ of the sum $\mathcal{E}(q_1 - q_2, Q_l^{0+}) \oplus \mathcal{E}(q_3, Q_3)$, and tight internal ellipsoidal approximation $\mathcal{E}(q_1 - q_2 + q_3, Q_l^{-})$ for the sum $\mathcal{E}(q_1 - q_2, Q_l^{0-}) \oplus \mathcal{E}(q_3, Q_3)$.

As a result, we get

$$\mathcal{E}(q_1 - q_2 + q_3, Q_l^{-}) \subseteq \mathcal{E}(q_1, Q_1)\dot{-}\mathcal{E}(q_2, Q_2) \oplus \mathcal{E}(q_3, Q_3) \subseteq \mathcal{E}(q_1 - q_2 + q_3, Q_l^{+})$$

and

$$\rho(\pm l \mid \mathcal{E}(q_1 - q_2 + q_3, Q_l^{-})) = \rho(\pm l \mid \mathcal{E}(q_1, Q_1)\dot{-}\mathcal{E}(q_2, Q_2) \oplus \mathcal{E}(q_3, Q_3)) = \rho(\pm l \mid \mathcal{E}(q_1 - q_2 + q_3, Q_l^{+})).$$

Running l over all unit vectors that are not bad directions, this translates to

$$\bigcup_{\langle l,l\rangle=1} \mathcal{E}(q_1 - q_2 + q_3, Q_l^{-}) = \mathcal{E}(q_1, Q_1)\dot{-}\mathcal{E}(q_2, Q_2) \oplus \mathcal{E}(q_3, Q_3) = \bigcap_{\langle l,l\rangle=1} \mathcal{E}(q_1 - q_2 + q_3, Q_l^{+}).$$

3.6 Geometric sum-difference

Given ellipsoids $\mathcal{E}(q_1, Q1)$, $\mathcal{E}(q_2, Q_2)$ and $\mathcal{E}(q_3, Q_3)$, it is possible to compute families of external and internal approximating ellipsoids for

$$\mathcal{E}(q_1, Q_1) \oplus \mathcal{E}(q_2, Q_2)\dot{-}\mathcal{E}(q_3, Q_3) \tag{3.17}$$

parametrized by direction l, if this set is nonempty ($\mathcal{E}(0, Q_3) \subseteq \mathcal{E}(0, Q_1) \oplus \mathcal{E}(0, Q_2)$).

First, using the result of Section 3.3, we obtain tight external $\mathcal{E}(q_1 + q_2, Q_l^{0+})$ and internal $\mathcal{E}(q_1 + q_2, Q_l^{0-})$ ellipsoidal approximations of the set $\mathcal{E}(q_1, Q_1) \oplus \mathcal{E}(q_2, Q_2)$. In order for the set (3.17) to be nonempty, inclusion $\mathcal{E}(0, Q_3) \subseteq \mathcal{E}(0, Q_l^{0+})$ must be true for any l. Note, however, that even if (3.17) is nonempty, it may be that $\mathcal{E}(0, Q_3) \not\subseteq \mathcal{E}(0, Q_l^{0-})$, then internal approximation for this direction does not exist.

Assuming that (3.17) is nonempty and $\mathcal{E}(0, Q_3) \subseteq \mathcal{E}(0, Q_l^{0-})$, the second step would be, using the results of section 2.2.3, to compute tight external ellipsoidal approximation $\mathcal{E}(q_1 + q_2 - q_3, Q_l^{+})$ of the difference $\mathcal{E}(q_1 + q_2, Q_l^{0+})\dot{-}\mathcal{E}(q_3, Q_3)$, which exists only if l is not a bad direction, and tight internal ellipsoidal approximation $\mathcal{E}(q_1 + q_2 - q_3, Q_l^{-})$ of the difference $\mathcal{E}(q_1 + q_2, Q_l^{0-})\dot{-}\mathcal{E}(q_3, Q_3)$, which exists only if l is not a bad direction for this difference.

If approximation $\mathcal{E}(q_1 + q_2 - q_3, Q_l^{+})$ exists, then

$$\mathcal{E}(q_1, Q_1) \oplus \mathcal{E}(q_2, Q_2)\dot{-}\mathcal{E}(q_3, Q_3) \subseteq \mathcal{E}(q_1 + q_2 - q_3, Q_l^{+})$$

and

$$\rho(\pm l \mid \mathcal{E}(q_1, Q_1) \oplus \mathcal{E}(q_2, Q_2)\dot{-}\mathcal{E}(q_3, Q_3)) = \rho(\pm l \mid \mathcal{E}(q_1 + q_2 - q_3, Q_l^{+})).$$

If approximation $\mathcal{E}(q_1 + q_2 - q_3, Q_l^-)$ exists, then

$$\mathcal{E}(q_1 + q_2 - q_3, Q_l^-) \subseteq \mathcal{E}(q_1, Q_1) \oplus \mathcal{E}(q_2, Q_2) \dot{-} \mathcal{E}(q_3, Q_3)$$

and

$$\rho(\pm l \mid \mathcal{E}(q_1 + q_2 - q_3, Q_l^-)) = \rho(\pm l \mid \mathcal{E}(q_1, Q_1) \oplus \mathcal{E}(q_2, Q_2) \dot{-} \mathcal{E}(q_3, Q_3)).$$

For any fixed direction l it may be the case that neither external nor internal tight ellipsoidal approximations exist.

4 Ellipsoidal method

Consider discrete-time linear system

$$x(t+1) = A(t)x(t) + B(t)u(t,x) + G(t)v(t), \tag{4.1}$$

in which $x(t) \in \mathbf{R}^n$ is the state, $u(t,x) \in \mathbf{R}^m$ is the control bounded by the ellipsoid $\mathcal{E}(p(t), P(t))$, $v(t) \in \mathbf{R}^d$ is disturbance bounded by ellipsoid $\mathcal{E}(q(t), Q(t))$, and matrices $A(t)$, $B(t)$, $G(t)$ are in $\mathbf{R}^{n\times n}$, $\mathbf{R}^{n\times m}$, $\mathbf{R}^{n\times d}$ respectively. Here we shall assume $A(t)$ to be nonsingular. The set of initial conditions at initial time t_0 is ellipsoid $\mathcal{E}(x_0, X_0)$.

Remark. The case when $A(t)$ is singular is described in [25]. The idea is to substitute $A(t)$ with the nonsingular $A_\delta(t) = A(t) + \delta U(t)W(t)$, in which $U(t)$ and $W(t)$ are obtained from the singular value decomposition

$$A(t) = U(t)\Sigma(t)W(t).$$

Parameter δ can be chosen based on the number of time steps for which the reach set must be computed and the required accuracy. The issue of inverting ill-conditioned matrices is also addressed in [25].

If matrix $Q(\cdot) = 0$, the system (4.1) becomes an ordinary affine system with known $v(\cdot) = q(\cdot)$. If matrix $G(\cdot) = 0$, the system reduces to a linear controlled system. In the absence of disturbance ($Q(\cdot) = 0$ or $G(\cdot) = 0$), $\overline{\mathcal{X}}_{CL}(t, t_0, \mathcal{E}(x_0, X_0)) = \underline{\mathcal{X}}_{CL}(t, t_0, \mathcal{E}(x_0, X_0)) = \mathcal{X}(t, t_0, \mathcal{E}(x_0, X_0))$, the reach set is as in Definition 2.1.

Maxmin and minmax CLRS $\overline{\mathcal{X}}_{CL}(t, t_0, \mathcal{E}(x_0, X_0)$ and $\underline{\mathcal{X}}_{CL}(t, t_0, \mathcal{E}(x_0, X_0)$, if nonempty, are symmetric convex and compact, with the center evolving in time according to

$$x_c(t+1) = A(t)x_c(t) + B(t)p(t) + G(t)v(t), \quad x_c(t_0) = x_0. \tag{4.2}$$

Fix some vector $l_0 \in \mathbf{R}^n$ and consider $l(t)$ that satisfies the discrete-time adjoint equation,[5]

$$l(t+1) = \left(A^T\right)^{-1}(t)l(t), \quad l(t_0) = l_0, \tag{4.3}$$

[5]Note that for (4.3) $A(t)$ must be invertible.

or, equivalently

$$l(t) = \Phi^T(t_0, t)l_0.$$

There exist tight external ellipsoids $\mathcal{E}(x_c(t), \overline{X}_l^+(t))$, $\mathcal{E}(x_c(t), \underline{X}_l^+(t))$ and tight internal ellipsoids $\mathcal{E}(x_c(t), \overline{X}_l^-(t))$, $\mathcal{E}(x_c(t), \underline{X}_l^-(t))$ such that

$$\mathcal{E}(x_c(t), \overline{X}_l^-(t)) \subseteq \overline{\mathcal{X}}_{CL}(t, t_0, \mathcal{E}(x_0, X_0)) \subseteq \mathcal{E}(x_c(t), \overline{X}_l^+(t)), \tag{4.4}$$

$$\rho(l(t) \mid \mathcal{E}(x_c(t), \overline{X}_l^-(t))) = \rho(l(t) \mid \overline{\mathcal{X}}_{CL}(t, t_0, \mathcal{E}(x_0, X_0))) = \rho(l(t) \mid \mathcal{E}(x_c(t), \overline{X}_l^+(t))). \tag{4.5}$$

and

$$\mathcal{E}(x_c(t), \underline{X}_l^-(t)) \subseteq \underline{\mathcal{X}}_{CL}(t, t_0, \mathcal{E}(x_0, X_0)) \subseteq \mathcal{E}(x_c(t), \underline{X}_l^+(t)), \tag{4.6}$$

$$\rho(l(t) \mid \mathcal{E}(x_c(t), \underline{X}_l^-(t))) = \rho(l(t) \mid \underline{\mathcal{X}}_{CL}(t, t_0, \mathcal{E}(x_0, X_0))) = \rho(l(t) \mid \mathcal{E}(x_c(t), \underline{X}_l^+(t))). \tag{4.7}$$

Remark. It is important to note that formulas (4.5) and (4.7) can hold only if for all time steps $\tau = t_0..t$ matrices $B(\tau)P(\tau)B^T(\tau)$ are not nondegenerate, which is, obviously not the case when control dimension m is strictly less than the state space dimension n. For each τ, $t_0 \leq \tau < t$, for which $B(\tau)P(\tau)B^T(\tau)$ is not of the full rank, it must be regularized as prescribed in [25], substituting $B(\tau)P(\tau)B^T(\tau)$ with $(B(\tau)P(\tau)B^T(\tau))_\alpha = B(\tau)P(\tau)B^T(\tau) + \alpha^2 I$, $\alpha > 0$, and the sign '=' in (4.5) and (4.7) should be changed to '≈'.

The shape matrix of the external ellipsoid for maxmin reach set is determined from

$$\hat{X}_l^+(t) = (1 + \overline{\pi}_l(t))A(t)\overline{X}_l^+(t)A^T(t) + \left(1 + \frac{1}{\overline{\pi}_l(t)}\right) B(t)P(t)B^T(t), \tag{4.8}$$

$$\overline{X}_l^+(t+1) = \left((\hat{X}_l^+(t))^{1/2} + \overline{S}_l(t)(G(t)Q(t)G^T(t))^{1/2}\right)^T \times \left((\hat{X}_l^+(t))^{1/2} + \overline{S}_l(t)(G(t)Q(t)G^T(t))^{1/2}\right), \tag{4.9}$$

$$\overline{X}_l^+(t_0) = X_0, \tag{4.10}$$

wherein

$$\overline{\pi}_l(t) = \frac{\langle l(t+1), B(t)P(t)B^T(t)l(t+1)\rangle^{1/2}}{\langle l(t), \overline{X}_l^+(t)l(t)\rangle^{1/2}},$$

and the orthogonal matrix $\overline{S}_l(t)$ is determined by the equation

$$\overline{S}_l(t)(G(t)Q(t)G^T(t))^{1/2}l(t+1) = \frac{\langle l(t+1), G(t)Q(t)G^T(t)l(t+1)\rangle^{1/2}}{\langle l(t+1), \hat{X}_l^+(t)l(t+1)\rangle^{1/2}}(\hat{X}_l^+(t))^{1/2}l(t+1).$$

Equation (4.9) is valid only if $\mathcal{E}(0, G(t)Q(t)G^T(t)) \subseteq \mathcal{E}(0, \hat{X}_l^+(t))$, otherwise the maxmin CLRS $\overline{\mathcal{X}}_{CL}(t, t_0, \mathcal{E}(x_0, X_0))$ is empty.

The shape matrix of the external ellipsoid for minmax reach set is determined from

$$\begin{aligned}
\check{X}_l^+(t) &= \Big((A(t)\underline{X}_l^+(t)A^T(t))^{1/2} + \underline{S}_l(t)(G(t)Q(t)G^T(t))^{1/2}\Big)^T \times \\
&\quad \Big((A(t)\underline{X}_l^+(t)A^T(t))^{1/2} + \underline{S}_l(t)(G(t)Q(t)G^T(t))^{1/2}\Big) && (4.11)\\
\underline{X}_l^+(t+1) &= (1+\underline{\pi}_l(t))\check{X}_l^+(t) + \left(1+\frac{1}{\underline{\pi}_l(t)}\right)B(t)P(t)B^T(t), && (4.12)\\
\underline{X}_l^+(t_0) &= X_0, && (4.13)
\end{aligned}$$

where

$$\underline{\pi}_l(t) = \frac{\langle l(t+1), B(t)P(t)B^T(t)l(t+1)\rangle^{1/2}}{\langle l(t+1), \check{X}_l^+(t)l(t+1)\rangle^{1/2}},$$

and $\underline{S}_l(t)$ is orthogonal matrix determined from the equation

$$\underline{S}_l(t)(G(t)Q(t)G^T(t))^{1/2}l(t+1) = \frac{\langle l(t+1), G(t)Q(t)G^T(t)l(t+1)\rangle^{1/2}}{\langle l(t), \underline{X}_l^+(t)l(t)\rangle^{1/2}}(A(t)\underline{X}_l^+(t)A^T(t))^{1/2}l(t+1).$$

Equations (4.11), (4.12) are valid only if $\mathcal{E}(0, G(t)Q(t)G^T(t)) \subseteq \mathcal{E}(0, A(t)\underline{X}_l^+(t)A^T(t))$, otherwise minmax CLRS $\underline{\mathcal{X}}_{CL}(t, t_0, \mathcal{E}(x_0, X_0))$ is empty.

The shape matrix of the internal ellipsoid for maxmin reach set is determined from

$$\begin{aligned}
\hat{X}_l^-(t) &= \Big((A(t)\overline{X}_l^-(t)A^T(t))^{1/2} + \overline{T}_l(t)(B(t)P(t)B^T(t))^{1/2}\Big)^T \times \\
&\quad \Big((A(t)\overline{X}_l^-(t)A^T(t))^{1/2} + \overline{T}_l(t)(B(t)P(t)B^T(t))^{1/2}\Big) && (4.14)\\
\overline{X}_l^-(t+1) &= (1+\overline{\eta}_l(t))\hat{X}_l^-(t) + \left(1+\frac{1}{\underline{\eta}_l(t)}\right)G(t)Q(t)G^T(t), && (4.15)\\
\overline{X}_l^-(t_0) &= X_0, && (4.16)
\end{aligned}$$

where

$$\overline{\eta}_l(t) = \frac{\langle l(t+1), G(t)Q(t)G^T(t)l(t+1)\rangle^{1/2}}{\langle l(t+1), \hat{X}_l^-(t)l(t+1)\rangle^{1/2}},$$

and $\overline{T}_l(t)$ is orthogonal matrix determined from the equation

$$\overline{T}_l(t)(B(t)P(t)B^T(t))^{1/2}l(t+1) = \frac{\langle l(t+1), B(t)P(t)B^T(t)l(t+1)\rangle^{1/2}}{\langle l(t), \overline{X}_l^-(t)l(t)\rangle^{1/2}}(A(t)\overline{X}_l^-(t)A^T(t))^{1/2}l(t+1).$$

Equation (4.15) is valid only if $\mathcal{E}(0, G(t)Q(t)G^T(t)) \subseteq \mathcal{E}(0, \hat{X}_l^-(t))$.

The shape matrix of the internal ellipsoid for the minmax reach set is determined by

$$\begin{aligned}
\breve{X}_l^-(t) &= (1+\underline{\eta}_l(t))A(t)\underline{X}_l^-(t)A^T(t) + \left(1+\frac{1}{\underline{\eta}_l(t)}\right)G(t)Q(t)G^T(t), && (4.17)\\
\underline{X}_l^-(t+1) &= \left((\breve{X}_l^-(t))^{1/2} + \underline{T}_l(t)(B(t)P(t)B^T(t))^{1/2}\right)^T \times \\
&\quad \left((\breve{X}_l^-(t))^{1/2} + \underline{T}_l(t)(B(t)P(t)B^T(t))^{1/2}\right), && (4.18)\\
\underline{X}_l^-(t_0) &= X_0, && (4.19)
\end{aligned}$$

wherein

$$\underline{\eta}_l(t) = \frac{\langle l(t+1), G(t)Q(t)G^T(t)l(t+1)\rangle^{1/2}}{\langle l(t), \underline{X}_l^-(t)l(t)\rangle^{1/2}},$$

and the orthogonal matrix $\underline{T}_l(t)$ is determined by the equation

$$\begin{aligned}
&\underline{T}_l(t)(B(t)P(t)B^T(t))^{1/2}l(t+1) = \\
&\frac{\langle l(t+1), B(t)P(t)B^T(t)l(t+1)\rangle^{1/2}}{\langle l(t+1), \breve{X}_l^-(t)l(t+1)\rangle^{1/2}}(\breve{X}_l^-(t))^{1/2}l(t+1).
\end{aligned}$$

Equations (4.17), (4.18) are valid only if $\mathcal{E}(0, G(t)Q(t)G^T(t)) \subseteq \mathcal{E}(0, A(t)\underline{X}_l^-(t)A^T(t))$.

The point where the external and the internal ellipsoids both touch the boundary of the maxmin CLRS is

$$x_l^+(t) = x_c(t) + \frac{\overline{X}_l^+(t)l(t)}{\langle l(t), \overline{X}_l^+(t)l(t)\rangle^{1/2}},$$

and the bounday point of minmax CLRS is

$$x_l^-(t) = x_c(t) + \frac{\overline{X}_l^-(t)l(t)}{\langle l(t), \overline{X}_l^-(t)l(t)\rangle^{1/2}}.$$

Points $x_l^{\pm}(t)$, $t \geq t_0$, form extremal trajectories. In order for the system to follow the extremal trajectory specified by some vector l_0, the initial state must be

$$x_l^0 = x_0 + \frac{X_0 l_0}{\langle l_0, X_0 l_0\rangle^{1/2}}. \qquad (4.20)$$

When there is no disturbance ($G(t) = 0$ or $Q(t) = 0$), $\overline{X}_l^+(t) = \underline{X}_l^+(t)$ and $\overline{X}_l^-(t) = \underline{X}_l^-(t)$, and the open-loop control that steers the system along the extremal trajectory defined by l_0 is

$$u_l(t) = p(t) + \frac{P(t)B^T(t)l(t+1)}{\langle l(t+1), B(t)P(t)B^T(t)l(t+1)\rangle^{1/2}}. \qquad (4.21)$$

Each choice of l_0 defines an external and internal approximation. If $\overline{\mathcal{X}}_{CL}(t, t_0, \mathcal{E}(x_0, X_0))$ is nonempty,

$$\bigcup_{\langle l_0, l_0\rangle = 1} \mathcal{E}(x_c(t), \overline{X}_l^-(t)) = \overline{\mathcal{X}}_{CL}(t, t_0, \mathcal{E}(x_0, X_0)) = \bigcap_{\langle l_0, l_0\rangle = 1} \mathcal{E}(x_c(t), \overline{X}_l^+(t)).$$

Similarly for $\underline{\mathcal{X}}_{CL}(t, t_0, \mathcal{E}(x_0, X_0))$,

$$\bigcup_{\langle l_0, l_0\rangle = 1} \mathcal{E}(x_c(t), \underline{X}_l^-(t)) = \underline{\mathcal{X}}_{CL}(t, t_0, \mathcal{E}(x_0, X_0)) = \bigcap_{\langle l_0, l_0\rangle = 1} \mathcal{E}(x_c(t), \underline{X}_l^+(t)).$$

Similarly, tight ellipsoidal approximations of maxmin and minmax CLBRS with terminating conditions $(t_1, \mathcal{E}(y_1, Y_1))$ can be obtained for those directions $l(t)$ satisfying

$$l(t) = \Phi^T(t_1, t) l_1, \tag{4.22}$$

with some fixed l_1, for which they exist.

With boundary conditions

$$y_c(t_1) = y_1, \quad \overline{Y}_l^+(t_1) = \overline{Y}_l^-(t_1) = \underline{Y}_l^+(t_1) = \underline{Y}_l^-(t_1) = Y_1, \tag{4.23}$$

external and internal ellipsoids for maxmin CLBRS $\overline{\mathcal{Y}}_{CL}(t_1, t, \mathcal{E}(y_1, Y_1))$ at time t, $\mathcal{E}(y_c(t), \overline{Y}_l^+(t))$ and $\mathcal{E}(y_c(t), \overline{Y}_l^-(t))$, are computed as external and internal ellipsoidal approximations of the geometric sum-difference

$$A^{-1}(t)\left(\left(\mathcal{E}(y_c(t+1), \overline{Y}_l^+(t+1)) \oplus B(t)\mathcal{E}(-p(t), P(t))\right) \dot{-} G(t)\mathcal{E}(-q(t), Q(t))\right) \tag{4.24}$$

and

$$A^{-1}(t)\left(\left(\mathcal{E}(y_c(t+1), \overline{Y}_l^-(t+1)) \oplus B(t)\mathcal{E}(-p(t), P(t))\right) \dot{-} G(t)\mathcal{E}(-q(t), Q(t))\right) \tag{4.25}$$

in direction $l(t)$ from 4.22. Section 3.6 describes the operation of geometric sum-difference for ellipsoids.

External and internal ellipsoids for minmax CLBRS $\underline{\mathcal{Y}}_{CL}(t_1, t, \mathcal{E}(y_1, Y_1))$ at time t, $\mathcal{E}(y_c(t), \underline{Y}_l^+(t))$ and $\mathcal{E}(y_c(t), \underline{Y}_l^-(t))$, are computed as external and internal ellipsoidal approximations of the geometric difference-sum

$$A^{-1}(t)\left(\left(\mathcal{E}(y_c(t+1), \underline{Y}_l^+(t+1)) \dot{-} G(t)\mathcal{E}(-q(t), Q(t))\right) \oplus B(t)\mathcal{E}(-p(t), P(t))\right) \tag{4.26}$$

and

$$A^{-1}(t)\left(\left(\mathcal{E}(y_c(t+1), \underline{Y}_l^-(t+1)) \dot{-} G(t)\mathcal{E}(-q(t), Q(t))\right) \oplus B(t)\mathcal{E}(-p(t), P(t))\right) \tag{4.27}$$

in direction $l(t)$ from 4.22. Section 3.5 describes the operation of geometric difference-sum for ellipsoids.

5 Control synthesis

The main application of the backward reachability is the control synthesis for steering the system to a given target set. For some combination of system dynamics, target set, control and disturbance constraints, it is possible to devise feedback control directly through dynamic programming, using (2.33) or (2.35). However, even for discrete-time systems when one does not solve the HJBI PDE, computation of max min or min max of the distance function is generally difficult. Therefore, we propose an ellipsoidal method that is adequate and efficient for linear systems.

Given system (4.1), target set defined by ellipsoid $\mathcal{E}(y_1, Y_1)$ and terminal time t_1, we want to find a closed-loop control that steers the system from some state y_0 at time $t_0 < t_1$ to $\mathcal{E}(y_1, Y_1)$ at t_1.

First we compute external ellipsoidal approximations $\mathcal{E}(y_c(t), \overline{Y}_l^+(t))$ of (4.24) and internal ellipsoidal approximations $\mathcal{E}(y_c(t), \overline{Y}_l^-(t))$ of (4.25) of the *maxmin* CLBRS for $t = t_0..(t_1-1)$, for different values of the parameter $l_1 \in \mathbf{R}^n$, as described in Section 3.6. If there exists an external ellipsoid $\mathcal{E}(y_c(t_0), \overline{Y}_l^+(t_0))$ such that $y_0 \notin \mathcal{E}(y_c(t_0), \overline{Y}_l^+(t_0))$, there is no closed-loop control that can guarantee taking the system from y_0 at t_0 to a state within $\mathcal{E}(y_1, Y_1)$ at t_1. On the other hand, if there exists an internal ellipsoid $\mathcal{E}(y_c(t_0), \overline{Y}_l^-(t_0))$ defined by the choice of l_1, such that $y_0 \in \mathcal{E}(y_c(t_0), \overline{Y}_l^-(t_0))$, such a control does exist.

We build the closed-loop control $u(t, y(t))$ so as to keep the system state $y(t)$ inside, if possible, or as close as we can, if not, to the internal approximating ellipsoid $\mathcal{E}(y_c(t), \overline{Y}_l^-(t))$ for $t = t_0..(t_1 - 1)$. The steps below describe control synthesis at time t.

1. Compute
$$\hat{y}(t+1) = A(t)y(t) + B(t)p(t) + G(t)v(t), \tag{5.1}$$
and set
$$\gamma(t+1) = \langle \hat{y}(t+1) - y_c(t+1), (\overline{Y}_l^-(t+1))^{-1}(\hat{y}(t+1) - y_c(t+1))\rangle. \tag{5.2}$$
If $\gamma(t+1) \leq 1$, then $\hat{y}(t+1) \in \mathcal{E}(y_c(t+1), \overline{Y}_l^-(t+1))$, and the control can be chosen as $u(t, y(t)) = p(t)$.

   Remark. Note that we compute $\hat{y}(t+1)$ with *known* $v(t)$. Recall definition 2.10: in maxmin case, at every time step we choose the control for the known disturbance.

2. Otherwise, if $\gamma(t+1) > 1$, $\hat{y}(t+1)$ is a boundary point of ellipsoid $\mathcal{E}(y_c(t+1), \gamma(t+1)\overline{Y}_l^-(t+1))$ corresponding to the direction $m(t+1) \in \mathbf{R}^n$,
$$m(t+1) = (\overline{Y}_l^-(t+1))^{-1}(\hat{y}(t+1) - y_c(t+1)).$$
In order to steer the system closer to the internal approximating ellipsoid, control $u(t, y(t))$ must act in the direction $-m(t+1)$.

3. Choose $u(t, y(t))$ so that the vector $B(t)u(t, y(t))$ is a boundary point of the ellipsoid $\mathcal{E}(B(t)p(t), B(t)P(t)B^T(t)) \subset \mathbf{R}^n$ in the direction $-m(t+1)$,
$$u(t, y(t)) = p(t) - \frac{P(t)B^T(t)m(t+1)}{\langle m(t+1), B(t)P(t)B^T(t)m(t+1)\rangle^{1/2}}.$$

To summarize,

$$u(t, y(t)) = p(t) - \begin{cases} 0, \text{ if } \gamma(t+1) \leq 1, \\ \frac{P(t)B^T(t)(\overline{Y}_l^-(t+1))^{-1}(\hat{y}(t+1)-y_c(t)+1)}{\langle(\overline{Y}_l^-(t+1))^{-1}(\hat{y}(t+1)-y_c(t+1)),B(t)P(t)B^T(t)(\overline{Y}_l^-(t+1))^{-1}(\hat{y}(t+1)-y_c(t+1))\rangle^{1/2}}, \text{ otherwise.} \end{cases} \tag{5.3}$$

Formula (5.3) gives *maxmin closed-loop* control.

We must again emphasize that maxmin feedback control strategy implies the knowledge of disturbance for the current time step. In reality we may not have such knowledge and should make a guess about the value of $v(t)$. An obvious choice for $v(t)$ is

$$v(t) = \arg \max_{v(t) \in \mathcal{E}(q(t), Q(t))} \gamma(t+1),$$

with $\gamma(t+1)$ defined in (5.1)-(5.2). This problem, however, is nonconvex and can have infinitely many solutions. The wrong guess of $v(t)$ for some time step t, $t_0 \leq t < t_1$, may result in the system state ending up *outside the maxmin CLBRS* $\overline{\mathcal{Y}}_{CL}(t_1, t+1, \mathcal{E}(y_1, Y_1))$, in which case there is no longer a closed-loop control strategy that *can guarantee reaching the target set* $\mathcal{E}(y_1, Y_1)$ *at time* t_1.[6]

The logical way out of this problem is to compute *minmax* closed-loop control $\underline{u}(t, y(t))$, $t = t_0..(t_1 - 1)$, instead. If $\underline{u}(t, y(t))$ exists, it guarantees that

$$A(t)\underline{\mathcal{Y}}_{CL}(t_1, t, \mathcal{E}(y_1, Y_1)) + B(t)\underline{u}(t, y(t)) \oplus G\mathcal{E}(q(t), Q(t)) \subseteq \underline{\mathcal{Y}}_{CL}(t_1, t+1, \mathcal{E}(y_1, Y_1)),$$

and it does exist provided that $\underline{\mathcal{Y}}_{CL}(t_1, t, \mathcal{E}(y_1, Y_1))$ is nonempty for $t = t_0..(t_1 - 1)$. Finding it, however, is difficult, and there appears to be no straight-forward way. We suggest computing minmax control as

$$u(t, y(t)) = \begin{cases} p(t), \text{ if } \langle y(t) - y_c(t), (\underline{Y}_l^-(t))^{-1}(y(t) - y_c(t))\rangle \leq 1, \\ p(t) - \frac{P(t)B^T(t)(\underline{Y}_l^-(t))^{-1}(y(t)-y_c(t))}{\langle(\underline{Y}_l^-(t))^{-1}(y(t)-y_c(t)),B(t)P(t)B^T(t)(\underline{Y}_l^-(t))^{-1}(y(t)-y_c(t))\rangle^{1/2}}, \text{ otherwise,} \end{cases} \tag{5.4}$$

where $\mathcal{E}(y_c(t), \underline{Y}_l^-(t))$ is the internal ellipsoidal approximation of (4.27) for $t = t_0..(t_1 - 1)$.

Remark. Note that if minmax CLBRS $\underline{\mathcal{Y}}_{CL}(t_1, t, \mathcal{E}(y_1, Y_1))$ is empty, which is often the case, there exist no minmax closed-loop control that guarantees reaching tha target set $\mathcal{E}(y_1, Y_1)$ at time step t_1.

6 Examples

6.1 Maxmin and minmax reach sets

We start with a simple example illustrating the concept of maxmin and minmax reach sets. Consider a continuous-time linear system

$$\dot{x}(s) = Ax(s) + Bu(s, x) + Gv(s), \tag{6.1}$$

[6]The target set $\mathcal{E}(y_1, Y_1)$ may still be reached at time t_1, but only if the disturbance allows.

where s represents continuous time; $x \in \mathbf{R}^2$; matrices $A = \begin{bmatrix} -0.1 & 1 \\ -2 & -0.1 \end{bmatrix}$, $B = \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}$, $G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are constant; control is bounded by constant box-valued constraints $\|u\|_\infty \leq 1$ for all $s > 0$; and so is the disturbance: $\|v\|_\infty \leq 1$ for all $s > 0$. The set of initial states is a unit box centered at $\begin{bmatrix} -4 \\ 4 \end{bmatrix}$: $\mathcal{X}_0 = \{x \in \mathbf{R}^2 \mid -5 \leq x_1 \leq -3,\ 3 \leq x_2 \leq 5\}$.

The sampling of (6.1) leads to a discrete-time linear system

$$x(t+1) = e^{A\Delta t}x(t) + \left(\int_0^{\Delta t} e^{A\Delta t}B\right) u(t,x) + \left(\int_0^{\Delta t} e^{A\Delta t}G\right) v(t), \tag{6.2}$$

where Δt is the sampling period.

For system (6.2) with given initial set, control and disturbance bounds, maxmin and minmax reach sets can be computed exactly via formulas (2.24), (2.19), (2.25), (2.20) and the polytope method from [6].

Figure 1 presents maxmin and minmax reach sets of the system (6.2) computed for the same termination time $T = 4$, and three different values of the sampling period $\Delta t = 0.4, 0.1$ and 0.04.[7] and three different values of the sampling period $\Delta t = 0.4, 0.1$ and 0.04. The three left plots depict the phase trajectories of the maxmin (blue), $\overline{\mathcal{X}}_{CL}(\tau, 0, \mathcal{X}_0)$ and the minmax (green), $\underline{\mathcal{X}}_{CL}(\tau, 0, \mathcal{X}_0)$, reach sets, $\tau = 0..t$; and the three right plots show $\overline{\mathcal{X}}_{CL}(t, 0, \mathcal{X}_0)$ and $\underline{\mathcal{X}}_{CL}(t, 0, \mathcal{X}_0)$, where $t = \frac{T}{\Delta t}$. This picture is a perfect illustration of the fact that $\underline{\mathcal{X}}_{CL}(\tau, 0, \mathcal{X}_0) \subseteq \overline{\mathcal{X}}_{CL}(\tau, , 0, \mathcal{X}_0)$ for $\tau = 0..t$, and as Δt gets smaller (the corrections happen more frequently), boundaries of maxmin and minmax reach sets get closer to each other.

Remark. For linear systems, as $\Delta t \to 0$, $\overline{\mathcal{X}}_{CL} = \underline{\mathcal{X}}_{CL}$, so in the continuous-time case there is no need to indicate whether CLRS is of maxmin or minmax type.

6.2 Steering to the target

The second example demonstrates how to steer the system to the target as described in Section 5 in given number of time steps.

Consider a linear system with disturbance

$$x(t+1) = \begin{bmatrix} 0 & 1 \\ -0.5 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t,x) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t), \tag{6.3}$$

where $x \in \mathbf{R}^2$, control u and disturbance v are restricted by constant bounds: $0 \leq u \leq 2$ and $-0.03 \leq v \leq 0.03$. Target set $\mathcal{Y}_1 = \mathcal{E}([1, -5]^T, I_{2\times 2})$ must be reached at $t_1 = 10$ starting at $t = 0$. The initial position of the system is $t_0 = 0$, $x_0 = \begin{bmatrix} 400 \\ 140 \end{bmatrix}$.

[7] T is given for the continuous-time system (6.1).

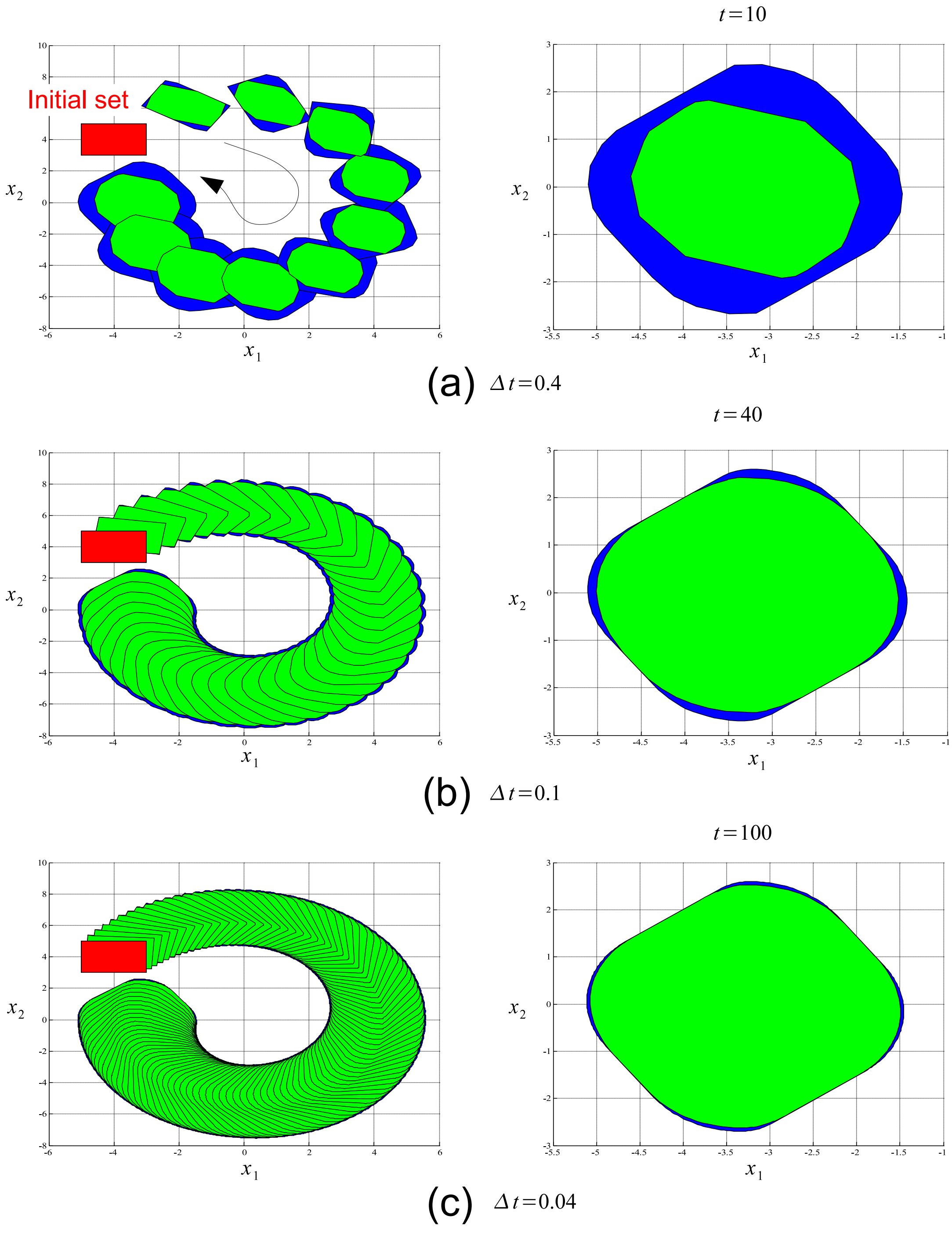

Figure 1: Maxmin (blue) and minmax (green) reach sets for system (6.2).

We start by arbitrarily fixing parameter $l_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and computing internal ellipsoidal approximation of the maxmin CLBRS for 10 time steps as described in Section 4. In this example, $x_0 \in \mathcal{E}(y_c(0), \overline{Y}_l^-(0))$ for the chosen l_1.[8] Figure 2 shows how this ellipsoidal approximation evolves from $t = 1$ to $t = t_1 = 10$.

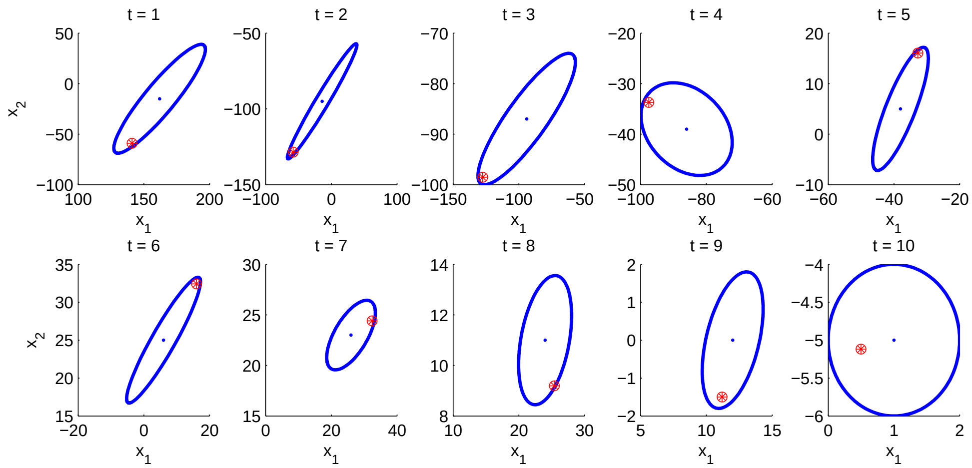

Figure 2: Internal ellipsoidal approximation of the maxmin CLBRS of system (6.3) computed for 10 time steps back. Closed-loop system trajectory is indicated by points in red.

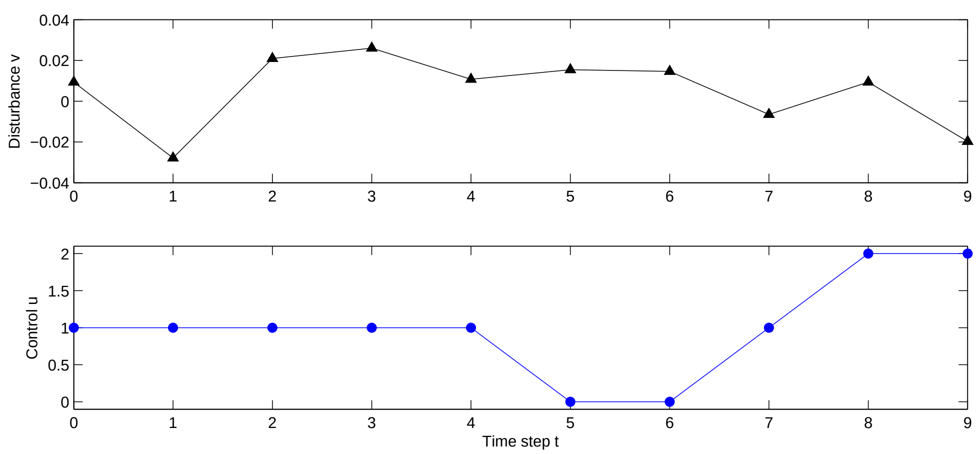

Figure 3: Random disturbance values (black) picked within given bounds, and control values (blue) computed via (5.3) that steer the system to the target set in 10 time steps.

Then, for every step $t = 0..9$ the control value is calculated from (5.3), while the disturbance values come from the outside world.[9] The resulting feedback control is displayed in Figure

[8]The general approach is to obtain internal ellipsoidal approximations for several values of l_1 and then select the one, for which the corresponding $\mathcal{E}(y_c(0), \overline{Y}_l^-(0))$ is the closest to x_0.

[9]For this particular example, the disturbance values were chosen randomly within their specified bounds.

3 together with the disturbance, for which it was computed. The aim of the control is to keep the system inside the chosen ellipsoidal approximation.

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