Congestion in the ACTM model

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***Abstract*— The equilibria of a simplified Asymmetric Cell Transmission Model (ACTM model) and their stability properties are investigated to reveal some patterns of congestion. These patterns appear to hold in empirical analysis of data for I-210W in Los Angeles.**

I. Introduction

Discrete macroscopic models are used for the empirical study of freeway traffic, propagation of congestion, and ramp metering. These models [6], [7], [8] begin with partial differential equations [5], [10] or directly with difference equations [2]. This paper analyzes a simplified version of the Asymmetric Cell Transmission Model (ACTM) [3], [4], which in turn is based on [2].

Section II presents the model. The freeway is divided into N sections. Each section is characterized by a single state variable, its density. Movement in a section is governed by its 'fundamental diagram'. If the density is below critical, vehicles move at free flow speed, if it is above critical, the section is congested and speed is lower.

Thus the state of a freeway with N sections obeys a N-dimensional nonlinear difference equation. When the exogenous demand—the profile of on-ramp and off-ramp flows—is constant, the state equation is time-invariant. Suppose the demand is feasible. The paper investigates the existence and stability properties of the equilibria of the state equation.

Section III shows that there is a unique uncongested equilibrium, Theorem 3.1. Usually there are several equilibria, but all of them share the same mainline flow pattern. In the special case that the demand is strictly feasible, i.e. all flows are strictly below capacity, the equilibrium is unique; it is also stable, Theorem 3.2. Lemma 3.2 explains how a bottleneck can be caused by on-ramp flows.

Section IV studies in detail the case in which there is only one non-zero on-ramp flow in the most downstream section. There are then exactly N equilibria, all stable. For each I, there is an equilibrium such that sections downstream of I are congested, those upstream of I are uncongested, Theorem 4.2.

Symbol	name	value	unit
F_i	capacity	20	veh/period
v_i	free flow speed	0.5	section/period
w_i	congestion wave speed	0.5/3	section/period
$\bar{n}_i$	jam density	160	veh/section
$n_{c,i}$	critical density	40	veh/section
β_i	split ratio	$\in [0,1]$	dimensionless
$\bar{\beta}_i$	complementary split ratio $= 1-\beta_i$	$\in [0,1]$	dimensionless
γ_i	on-ramp blending factor in i	$\in [0,1]$	dimensionless
f_i	flow leaving section i	variable	veh/period
s_i, r_i	off-ramp, on-ramp flow in section i	variable	veh/period
n_i	number of vehicles in section i	variable	veh/section

TABLE I
Model parameters and variables.

Section V fits the model to a 14-mile section of I-210W in Los Angeles.

II. The ACTM Model

Figure 1 shows the freeway divided into N sections. Vehicles move from right to left. Section i is upstream of section $i-1$. There are two boundary conditions. Free flow prevails downstream of section 0; upstream of the freeway is an "on-ramp" with an inflow of r_N. The flow accepted by section $N-1$ is f_N veh/time step. The cumulative difference leads to a queue of size $n_N(k)$ at time step k.

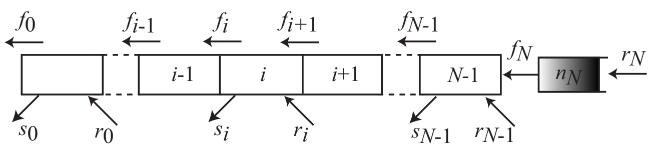

Fig. 1. The freeway has N sections. Each section has at most one on- and one off-ramp.

Table I lists the model variables and parameters with plausible values. The length of all sections is normalized to 1 by absorbing differences in length in the speeds v_i, w_i. To be physically meaningful one must have $0 < v_i, w_i < 1$. The parameter values in the table correspond to the fundamental diagram of Figure 2. Its triangular form incorporates the assumption that will be used frequently:

$$F_i = \bar{\beta}_i \, v_i \, n_{c,i} = w_{i-1} \, (\bar{n}_{i-1} - n_{c,i-1}). \tag{1}$$

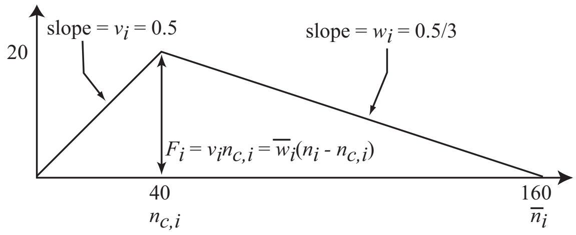

Fig. 2. The fundamental diagram is characterized by the maximum flow F and speeds v, w.

Offramp flows are modeled as a portion $\beta_i(k)$ of the total flow leaving the section:

$$s_i(k) = \beta_i(k)(s_i(k) + f_i(k))$$

We will assume constant split ratios β_i throughout. Defining $\bar{\beta}_i = 1 - \beta_i$, the ACTM model equations become, for $k \geq 0$,

$$\begin{aligned} n_i(k+1) &= n_i(k) - f_i(k)/\bar{\beta}_i + f_{i+1}(k) + r_i(k), \\ &\quad 0 \leq i \leq N-1, \quad (2) \\ f_i(k) &= \min\{\bar{\beta}_i\, v_i[n_i(k) + \gamma_i r_i(k)], \\ &\quad w_{i-1}[\bar{n}_{i-1} - n_{i-1}(k) - \gamma_{i-1} r_{i-1}(k)], F_i\}, \\ &\quad 1 \leq i \leq N, \quad (3) \\ f_0(k) &= \min\{\bar{\beta}_0\, v_0[n_0(k) + \gamma_0 r_0(k)], F_0\}, \quad (4) \\ n_N(k+1) &= n_N(k) - f_N(k) + r_N(k). \quad (5) \end{aligned}$$

Flow conservation is expressed by (2) and (5); the latter gives the change in queue size n_N at the upstream ramp. The flow $f_i(k)$ is governed by the fundamental diagram (3): $\bar{\beta}_i v_i[n_i(k)+\gamma_i r_i(k)]$ is the number of vehicles that can move from section i to $i-1$ in period k; $w_{i-1}[\bar{n}_{i-1} - n_{i-1}(k) - \gamma_{i-1} r_{i-1}(k)]$ is the number that $i-1$ can accept; and F_i is the capacity of section i. Equation (4) indicates there is no congestion downstream of section 0.

The *state* of the system is the N-dimensional vector $n(k) = (n_0(k), \cdots, n_{N-1}(k))$. The queue size $n_N(k)$ is *not* included in the state.

III. Equilibria and Bottlenecks

$\gamma_i \in [0,1]$ reflects the relative position of the on-ramp in section i [3], [4]. For simplicity we assume $\gamma_i = 0$, indicating that the on-ramp is at the beginning of each section as in Figure 1. Given a non-negative constant demand or ramp flow pattern, $\mathcal{F} = \{r_0, s_0, \cdots, r_{N-1}, s_{N-1}, r_N\}$, recursively calculate $f_N, \cdots, f_0$ from (6) and (7).

$$\begin{aligned} f_N &= r_N, \quad (6) \\ f_i &= \bar{\beta}_i\,(f_{i+1} + r_i), \quad 0 \leq i \leq N-1. \quad (7) \end{aligned}$$

Say that $\mathcal{F}$ is *feasible* if $0 \leq f_i \leq F_i$, $0 \leq i \leq N$; and $\mathcal{F}$ is *strictly feasible* if $0 \leq f_i < F_i$, $0 \leq i \leq N$.

A state vector $(n_1, \cdots, n_{N-1})$ is an *equilibrium*, if $n_i(k) \equiv n_i$, $1 \leq i \leq N-1$ is a solution of (2)-(4), i.e.

$$\begin{aligned} f_i &= \min\{\bar{\beta}_i\, v_i\, n_i, w_{i-1}(\bar{n}_{i-1} - n_{i-1}), F_i\}, \\ &\quad 1 \leq i \leq N-1, \quad (8) \\ f_0 &= \min\{\bar{\beta}_0\, v_0\, n_0, F_0\}. \quad (9) \end{aligned}$$

At equilibrium n, section i is *uncongested* if $0 \leq n_i \leq n_c, 0 \leq i \leq N-1$; it is *congested* if $n_i > n_c, 0 \leq i \leq N-1$; n is *uncongested* if all sections are uncongested.

Theorem 3.1: A feasible flow pattern $\mathcal{F}$ has a unique uncongested equilibrium, n^e. Moreover, all equilibria, whether congested or uncongested, share the same flow pattern.

Proof. The second assertion is immediate since (6), (7) yield a unique flow in each section. We now prove existence and uniqueness of the uncongested equilibrium.

Existence: Define $n_i^e = (\bar{\beta}_i\, v_i)^{-1} f_i$, $0 \leq i \leq N-1$. Then $n_i(k) \equiv n_i^e$ satisfies (2), because (2) is equivalent to (7). Next, because $0 \leq f_i \leq F_i$ and $F_i = \bar{\beta}_i v_i n_{c,i}$ (see (1)), $n_i^e = (\bar{\beta}_i\, v_i)^{-1} f_i \leq (\bar{\beta}_i\, v_i)^{-1} F_i = n_{c,i}$. So n^e is uncongested.

It remains to prove that n^e is an equilibrium, i.e. satisfies (8). First, by construction, $f_i = \bar{\beta}_i\, v_i\, n_i^e$. Second, since $\mathcal{F}$ is feasible, $f_i \leq F_i$. Lastly, since n^e is uncongested, $n_{i-1}^e \leq n_{c,i-1}$, and so (from (1)),

$$w_{i-1}(\bar{n}_{i-1} - n_{i-1}^e) \geq w_{i-1}(\bar{n}_{i-1} - n_{c,i-1}) = F_i.$$

Hence $f_i \leq w_{i-1}(\bar{n}_{i-1} - n_{i-1}^e)$. This proves (8).

Uniqueness: Suppose $\{0 \leq n_i \leq n_{c,i}; 0 \leq i \leq N-1\}$ is an equilibrium, i.e. satisfies (8)-(9). Since $n_i \leq n_{c,i}$, $\bar{\beta}_i\, v_i\, n_i \leq \bar{\beta}_i\, v_i\, n_{c,i} = F_i$, and so (8) simplifies to

$$f_i = \min\{\bar{\beta}_i v_i n_i, w_{i-1}(\bar{n}_{i-1} - n_{i-1})\}.$$

If $f_i \neq \bar{\beta}_i v_i n_i$, it must be that $\bar{\beta}_i v_i n_i > w_{i-1}(\bar{n}_{i-1} - n_{i-1}) \geq w_{i-1}(\bar{n}_{i-1} - n_{c,i-1}) = F_i$. This contradicts $\bar{\beta}_i\, v_i\, n_i \leq F_i$, so f_i must equal $\bar{\beta}_i v_i n_i$. □

To reduce the notational burden we assume henceforth that all sections are *identical*, and we drop the suffix i from the parameters $F_i, v_i, w_i, n_{c,i}, \bar{n}_i$.

Lemma 3.1: Let $\mathcal{F}$ be a strictly feasible flow pattern and let $(n_0, \cdots, n_{N-1})$ be the corresponding unique uncongested equilibrium. Then $n_c - n_i > 0$ for all i, and there is $\Delta > 0$ such that if the initial state $n(0)$ satisfies $n_i(0) - n_i < \Delta$, for all i, the state trajectory $n(k), k \geq 0$, of the original system (2)-(4) satisfies

$$n_i(k) < n_c, \quad 0 \leq i \leq N-1, k \geq 0, \quad (10)$$

and

$$\lim_{k \leftarrow \infty} n_i(k) = n_i, \quad 0 \leq i \leq N-1. \quad (11)$$

That is, the uncongested equilibrium of a strictly feasible flow is (locally) asymptotically stable.

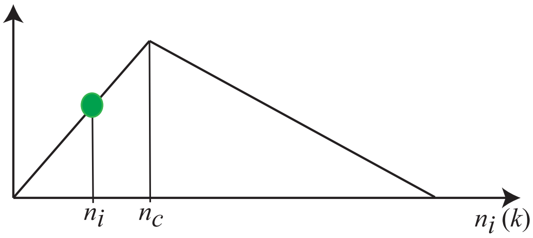

Fig. 3. Illustration for Lemma 3.1.

Proof See Figure 3. Since $n_i = (\bar{\beta}_i v)^{-1} f_i < (\bar{\beta}_i v)^{-1} F = n_c$, $n_c - n_i > 0$.

Take any initial state $n(0) = (n_0(0), \cdots, n_{N-1}(0))$ and let $n(k) = \{n_i(k)\}$ be the resulting trajectory. Assume for now that (10) holds. Then from (3) one has $f_i(k) = \bar{\beta}_i v n_i(k)$, and substituting this into (2) gives for all i, k,

$$n_i(k+1) = (1-v)n_i(k) + \bar{\beta}_{i+1} v n_{i+1}(k) + r_i.$$

Write $n_i(k) = n_i + \delta_i(k)$. (The assumption (10) is then equivalent to the assumption $\delta_i(k) < n_i - n_c$.) Substituting in the previous equation gives

$$n_i+\delta_i(k+1) = (1-v)(n_i+\delta_i(k)) + \bar{\beta}_{i+1} v(n_{i+1}+\delta_{i+1}(k)) + r_i,$$

or

$$\delta_i(k+1) = (1-v)\delta_i(k) + \bar{\beta}_{i+1} v \delta_{i+1}(k) - v n_i + \bar{\beta}_{i+1} v n_{i+1} + r_i,$$

and since $v n_i = \bar{\beta}_{i+1} v n_{i+1} + r_i$ by (7), one has

$$\delta_i(k+1) = (1-v)\delta_i(k) + \bar{\beta}_{i+1} v \delta_{i+1}(k), \quad 0 \le i \le N-1, k \ge 0,$$

or in matrix form:

$$\begin{bmatrix} \delta_0(k+1) \\ \delta_1(k+1) \\ \cdot \\ \delta_N(k+1) \end{bmatrix} = \begin{bmatrix} 1-v & \bar{\beta}_1 v & 0 & \cdots & 0 \\ 0 & 1-v & \bar{\beta}_2 v & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 1-v \end{bmatrix} \times \begin{bmatrix} \delta_0(k) \\ \delta_1(k) \\ \cdot \\ \delta_N(k) \end{bmatrix}. \qquad (12)$$

Denote the matrix in (12) by A, and the vector by $\delta(k)$. Then

$$\delta(k) = A^k \delta(0), \quad k \ge 0.$$

Let $A^k = \{a_{ij}(k); 0 \le i, j \le N\}$. The entries $a_{ij}(k)$ are non-negative and uniformly bounded by some number, say α. Hence $\delta(k) = A^k\delta(0)$ satisfies

$$\delta_i(k) = \sum_j a_{ij}(k)\delta_j(0) < N\alpha \max_j |\delta_j(0)|.$$

Select Δ so that $N\alpha\Delta = \min_i\{n_c - n_i\}$. Then

$$\delta_i(k) < \min_i\{n_c - n_i\} \le (n_c - n_i),$$

or

$$n_i + \delta_i(k) \le n_c, \quad 0 \le i \le N-1, k \ge 0.$$

But then $n_i(k) = n_i + \delta_i(k)$ is indeed the solution to (2)-(4) and $n_i(k) < n_c$, proving (10). Moreover, as all eigenvalues of A equal $(1-v)$, $A^k \to 0$, as $k \to \infty$, from which (11) follows. □

Let $\mathcal{F}$ be a feasible flow pattern. Let $\{f_i\}$ be the resulting flows, $0 \le f_i \le F$. Let $n = (n_0, \cdots, n_{N-1})$ be *any* equilibrium.

Lemma 3.2: Suppose at the equilibrium n, section $i-1$ is uncongested and sections $i, \cdots, i+j$ are congested. Then

$$f_i = F, r_i > 0, f_{i+1} = \bar{\beta}_i^{-1}F - r_i < F, \\ \text{and } f_{i+1} < F, \cdots, f_{i+j+1} < F. \qquad (13)$$

Proof. Since $n_{i-1} \le n_c$, $F + w(n_c - n_{i-1}) \ge F$; hence from (8)

$$f_i = \min\{\bar{\beta}_i v n_i, F\}.$$

Since $n_i > n_c$, one has $\bar{\beta}_i v n_i > F$; since $\mathcal{F}$ is feasible, $F \ge f_i$. Hence, $f_i = F$.

Next, as $n_i > n_c$, $F + w(n_c - n_i) < F$, and so from (8)

$$f_{i+1} = \min\{\bar{\beta}_{i+1} v n_{i+1}, F + w(n_c - n_i), F\} < F$$

Then from (7),

$$r_i = \bar{\beta}_i^{-1} f_i - f_{i+1} = \bar{\beta}_i^{-1}F - f_{i+1} > F - f_{i+1} > 0$$

which shows $r_i > 0$ and $f_{i+1} = \bar{\beta}_i^{-1}F - r_i$.

Lastly, if $n_{i+k} > n_c$, $f_{i+k+1} = \min\{\bar{\beta}_{i+k+1} v n_{i+k+1}, F + w[n_c - n_{i+k}, F\} < F$, so the remaining assertion follows. □

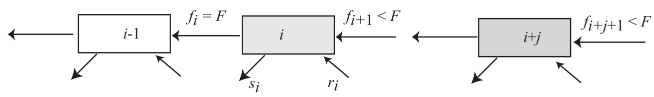

Fig. 4. Illustration for Lemma 3.2: section $i-1$ is uncongested and sections $i, \cdots, i+j$ are congested.

Figure 4 illustrates Lemma 3.2. Section $i-1$ is uncongested; sections $i, \cdots, i+j$ are congested. Then the flow out of the most downstream congested section must equal capacity, and the flow into all the congested sections must be below capacity. It is customary to call section i a *bottleneck* and F the *bottleneck capacity*. Since the on-ramp flow in the bottleneck section is $r_i > 0$, one might say that the bottleneck is *caused* by the on-ramp flow.

Since all freeway sections are identical, a bottleneck cannot be caused by a capacity drop. When the demand is time-varying the flow of vehicles exiting at an off-ramp may temporarily exceed the ramp capacity, causing a 'diverge' bottleneck [1]. A diverge bottleneck cannot occur in the stationary demand conditions analyzed here.

Whether a section is uncongested or congested is equivalent to whether vehicles move at or below free flow speed. Thus a bottleneck section is also characterized by the feature that in it vehicles move below free flow speed, but in the section downstream from it vehicles move at free flow. This feature is used in PeMS [9] to find bottlenecks.

The next result strengthens Lemma 3.1.

Theorem 3.2: A strictly feasible flow $\mathcal{F}$ has a unique equilibrium. Moreover, this equilibrium is uncongested and asymptotically stable.

Proof. If the equilibrium $(n_0, \cdots, n_{N-1})$ is uncongested, the result follows from Lemma 3.1. Suppose the equilibrium is not uncongested, so there is at least one congested section. There are two cases to consider. In the first case, section 0 is congested. So $f_0 = \min\{\bar{\beta}_0 v n_0, F\} = \bar{\beta}_0 v n_0$, since $f_0 < F$. But then $\bar{\beta}_0 v n_0 = f_0 < F$ by strict feasibility, and so $n_0 < n_c$, which means section 0 is not congested.

In the remaining case, there must exist a pair of adjacent sections $i-1, i$ with $n_{i-1} \leq n_c < n_i$, and so, by Lemma 1, $f_i = F$, which contradicts strict feasibility of $\mathcal{F}$. □

Conjecture. The equilibrium above is *globally* asymptotically stable.

IV. Congestion Patterns

Thus a strictly feasible flow has a unique equilibrium, which must be uncongested and stable. On the other hand, by Theorem 3.1, every feasible flow has a unique uncongested equilibrium. This raises the question: Does a feasible flow have an equilibrium in which some sections are congested, and can such an equilibrium be stable? The answer,"yes," is explored in the context of the following example.

Example. The capacity is F for all links. The upstream inflow is $r_N = F - r < F$; the inflow in section 0 is $r_0 = r$; all other on- and off-ramp flows are 0. Hence, from (7), (6), $f_i = F - r, 1 \leq i \leq N$, and $f_0 = f_1 + r = F$.

Figure 5 illustrates the example for the numerical values $F = 20, r = 8$.

For the example, the model (2)-(5) simplifies:

$$\begin{aligned}
n_0(k+1) &= n_0(k) - f_0(k) + r + f_1(k), && (14)\\
n_i(k+1) &= n_i(k) - f_i(k) + f_{i+1}(k), &&\\
& \qquad 1 \leq i \leq N-1, && (15)\\
n_N(k+1) &= n_N(k) - f_N(k) + F - r, && (16)\\
f_0(k) &= \min\{v n_0(k), F\}, && (17)\\
f_i(k) &= \min\{v n_i(k), F - w[n_{i-1}(k) - n_c], F\}, &&\\
& \qquad 1 \leq i \leq N. && (18)
\end{aligned}$$

We first verify that

$$n_i(k) \equiv n^e = \{n_c + w^{-1} r, \quad 0 \leq i \leq N-1\} \qquad (19)$$

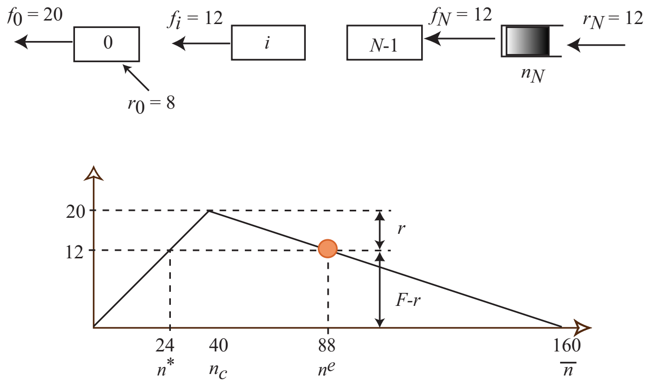

Fig. 5. Illustration for Example.

is an equilibrium, i.e. satisfies (17)-(18). Since $n_i^e > n_c$ and $\bar{\beta}_i = 1$,

$$\begin{aligned}
&\min\{\bar{\beta}_i v n_i^e, F - w(n_{i-1}^e - n_c)), F\}\\
&= F - w(n_{i-1}^e - n_c) = F - r = f_i, \quad 1 \leq i \leq N-1,
\end{aligned}$$

so (18) holds, and since $n_0^e > n_c$, $\min\{\bar{\beta}_0 v n_0^e, F\} = F = f_0$, so (17) holds.

The following proposition contrasts with Lemma 3.1.

Lemma 4.1: There exists a $\Delta > 0$ such that if the initial state $|n_i(0) - n^e| < \Delta$ for all i, the trajectory $\{n_i(k)\}$ satisfies $n_i(k) > n_c$, for all k, and

$$\lim_{k \to \infty} n_i(k) = n_i^e, \quad 0 \leq i \leq N-1. \qquad (20)$$

Proof. Assuming all sections (including N) to be congested, and defining $\delta_i(k) = n_i(k) - n^e$, the system dynamics can be written in matrix form:

$$\begin{bmatrix} \delta_0(k+1) \\ \delta_1(k+1) \\ \cdot \\ \delta_{N-1}(k+1) \end{bmatrix} = \begin{bmatrix} 1-w & 0 & 0 & \cdots & 0 \\ w & 1-w & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & w & 1-w \end{bmatrix} \times \begin{bmatrix} \delta_0(k) \\ \delta_1(k) \\ \cdot \\ \delta_{N-1}(k) \end{bmatrix} \qquad (21)$$

and

$$\delta_N(k+1) = \delta_N(k) + w \delta_{N-1}(k) \qquad (22)$$

which may be contrasted with (12). A similar proof to that of Lemma 3.1 applies as long as the system remains in the congested region. As sections 0 through $N-1$ are governed by the asymptotically stable linear system (21), we need only ensure that $\delta_N(k)$ remains congested, i.e. larger than $n_c - n^e$. From (22),

$$\begin{aligned}
\delta_N(k) = &\ \delta_N(0) \ + \ w \sum_{\kappa=0}^{k-1} \delta_{N-1}(\kappa)\\
\geq &\ \delta_N(0) \ - \ w \sum_{\kappa=0}^{k-1} |\delta_{N-1}(\kappa)| \qquad (23)
\end{aligned}$$

Let A denote the matrix in (21). Then

$$\delta(k) \;=\; A^k\,\delta(0)$$

It can be shown that

$$||A^k|| \leq M|\lambda(A)|^k$$

where M is some positive number and $\lambda(A)$ is the largest eigenvalue of A, which in this case equals $1-w$. Then

$$\begin{aligned} |\delta_{N-1}(k)| &\leq ||\delta(k)|| \\ &\leq ||A^k||\;||\delta(0)|| \\ &\leq ||\delta(0)||\;M\;(1-w)^k \end{aligned}$$

Thus,

$$\begin{aligned} w\sum_{\kappa=0}^{k-1}|\delta_{N-1}(\kappa)| &\leq ||\delta(0)||\;M\;w\sum_{\kappa=0}^{k-1}(1-w)^\kappa \\ &= ||\delta(0)||\;M\;\left(1-(1-w)^k\right) \\ &= ||\delta(0)||\;Q \end{aligned}$$

where we have defined $Q = M\left(1-(1-w)^k\right)$. Substituting this relation into (23) and using $|\delta_i(0)| < \Delta$,

$$\begin{aligned} \delta_N(k) &\geq \delta_N(0)\;-\;||\delta(0)||\;Q(k) \\ &> -\Delta\;-\;\Delta\;Q \\ &= -(1+Q)\Delta \end{aligned}$$

Selecting $\Delta \leq (n^e - n_c)/(1+Q)$ then implies $\delta_N(k) > n_c - n^e$, as required. □

Note that it cannot be said that n^e is *asymptotically stable* because the lemma requires $|n_N(0) - n^e| < \Delta$, but only guarantees convergence for sections 0 through $N-1$. The equilibrium is *stable* however since $n_N(k)$ remains in an arbitrarily small vicinity of the equilibrium.

A strictly feasible flow pattern by Theorem 3.2 has a unique uncongested equilibrium, which must be stable. On the other hand, Lemma 4.1 shows a stable equilibrium in which *every section is congested*. Observe that in the numerical example of Figure 5, the free flow speed is 60 mph; the speed at the equilibrium n^e is 8.2 mph. Observe, too, that the flows in each section in both equilibria are *identical*. This belies the commonly voiced assertion that a congested freeway indicates demand exceeding capacity.

From the fundamental diagram in Figure 5 we can immediately guess at other equilibria $n^* = \{n_i^*\}$ of the form

$$n_i^* = v^{-1}(F-r),\;1\leq i\leq N;\;n_0^*\in[n_c, n_c+w^{-1}r), \quad (24)$$

in which section 0 is congested, but the rest are not.

Figure 6 shows the relationship between n_0^* and n_1^*. We first verify that n^* is indeed an equilibrium, i.e. satisfies (17)-(18). Since $\bar{\beta}_i\,v\,n_i^* < F$, $\bar{\beta}_i = 1$, and $n_{i-1}^* \leq n_c$,

$$\begin{aligned} \min\{\bar{\beta}_i\,v\,n_i^*, w(\bar{n}-n_{i-1}^*), F\} \;&=\; vn_i^* = F-r = f_i, \\ &\qquad 1\leq i\leq N-1, \end{aligned}$$

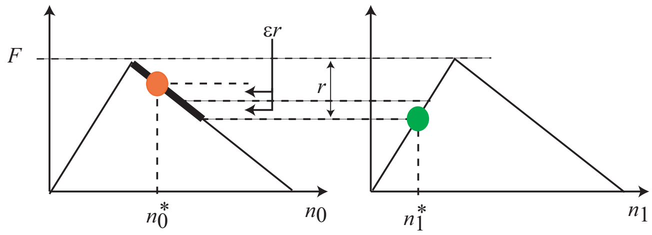

Fig. 6. Illustration of n_0^*, n_1^*

so (18) holds, and since $vn_0^* \geq vn_c = F$, $\min\{vn_0^*, F\} = F = f_0$, so (17) holds. Thus n^* is indeed an equilibrium.

Now consider a particular equilibrium, $n_0^* = n_c + [1-2\epsilon]w^{-1}r$ for some $0<\epsilon<1/2$ and $n_i^* = v^{-1}(F-r), i\geq 1$. We investigate the stability of n^*.

Lemma 4.2: n^* is a (locally) stable equilibrium: If $|n_0(0) - n_0^*| < (1-v)\epsilon w^{-1}r$, then $|n_0(k) - n_0^*| < \epsilon w^{-1}r$; and if $|n_i(0) - n_i^*|$ is sufficiently small, then $n_i(k)$ converges to n_i^*, $1\leq i\leq N-1$.

Proof. We look for solutions of (14)-(18) in which

$$\begin{aligned} n_0(k) &< n_0^* + \epsilon w^{-1}r = n_c + [1-\epsilon]w^{-1}r, \\ n_i(k) &< n_i^* + \epsilon v^{-1}r,\;1\leq i\leq N. \end{aligned} \quad (25)$$

We evaluate the expressions (17), (18) for such a solution. First, from (17),

$$f_0(k) = \min\{vn_0(k), F\} = \begin{cases} F, & \text{if } n_0(k) > n_c \\ vn_0(k) & \text{if } n_0(k)\leq n_c \end{cases} \quad (26)$$

Second, from (18), for $i=1$,

$$f_1(k) = \min\{vn_1(k), F - w(n_0(k)-n_c), F\}. \quad (27)$$

From (25)

$$vn_1(k) < v[n_1^* + \epsilon v^{-1}r] = F - [1-\epsilon]r$$

and

$$F - w(n_0(k)-n_c) > F - [1-\epsilon]r.$$

Substituting the two previous inequalities into (27) gives

$$f_1(k) = vn_1(k). \quad (28)$$

Third, from (25), for $i>1$,

$$\begin{aligned} v(n_{i-1}(k)-n_c) &= v(n_{i-1}(k)-n_i^*) + v(n_i^*-n_c) \\ &< -[1-\epsilon]r < 0, \end{aligned}$$

so that $F - w(n_{i-1}(k)-n_c) > F$. Hence from (18), for $i>1$,

$$f_i(k) = \min\{vn_i(k), F - w(n_{i-1}(k)-n_c), F\} = vn_i(k). \quad (29)$$

Substitution from (26)-(29) into (14)-(15) yields the system of equations

$$\begin{aligned} n_0(k+1) &= n_0(k) - \min\{vn_0(k), F\} \\ &+ r + vn_1(k), \qquad (30) \\ n_i(k+1) &= [1-v]n_i(k) + vn_{i+1}(k), \\ & \quad 1 \leq i \leq N-1, \qquad (31) \\ n_N(k+1) &= [1-v]n_N(k) + F - r. \qquad (32) \end{aligned}$$

Observe that (31)-(32) form a 'closed' system, with a stable equilibrium given by

$$\lim_{k\to\infty} n_i(k) = n_i^* = v^{-1}(F-r), \quad 1 \leq i \leq N. \qquad (33)$$

This convergence is exponentially fast; consequently there exists $\Delta > 0$ such that if $|n_i(0) - n_i^*| < \Delta$, $1 \leq i \leq N$, then

$$n_i(k) < n_i^* + \epsilon v^{-1} r, \quad 1 \leq i \leq N-1, k \geq 0,$$

and

$$\sum_{k=0}^{\infty} |n_1(k) - n_1^*| < \epsilon w^{-1} r. \qquad (34)$$

Let $\delta_0(k) = n_0(k) - n_0^*$, so

$$\begin{aligned} n_0(k) &= \delta_0(k) + n_0^* \\ &= \delta_0(k) + n_c + [1-2\epsilon]w^{-1}r \\ &= \delta_0(k) + v^{-1}F + [1-2\epsilon]w^{-1}r, \\ vn_0(k) &= v\delta_0(k) + F + v[1-2\epsilon]w^{-1}r \\ &= v\delta_0(k) + F + v\eta. \end{aligned}$$

wherein $\eta = (1-2\epsilon)w^{-1}r = n_0^* - n_c$. Hence

$$\min\{vn_0(k), F\} = \begin{cases} F, \text{ if } \delta_0(k) + \eta > 0 \\ F + v[\delta_0(k) + \eta], \text{otherwise} \end{cases}$$

Upon substituting this expression into (30) one gets

$$\delta_0(k+1) = \begin{cases} (1-v)\delta_0(k) + v[n_1(k) - n_1^*] - v\eta \\ \quad \text{if } \delta_0(k) + \eta \leq 0, \\ \delta_0(k) + v[n_1(k) - n_1^*] \\ \quad \text{if } \delta_0(k) + \eta > 0 \end{cases}, \qquad (35)$$

From (35)

$$\delta_0(k+1) \leq |\delta_0(k)| + v|n_1(k) - n_1^*|,$$

which, together with (34), gives

$$|\delta_0(k)| \leq |\delta_0(0)| + \sum_{k=0}^{\infty} v|n_1(k) - n_1^*| < |\delta_0(0)| + v\epsilon w^{-1} r.$$

Hence, if $|\delta_0(0)| < (1-v)\epsilon w^{-1} r$, (25) is satisfied. □

The equilibrium n^* is *not* asymptotically stable because $n_0(k)$ may not converge to n_0^*. Indeed we now show that $\delta_0(k)$ given by (35) converges to one of two possible limits. Let $K = \{0 \leq k_1 < k_2 < \cdots\}$ be the set of all k for which $\delta_0(k) + \eta \leq 0$, so the first expression in (35) holds. Then (35) can be rewritten as

$$\delta_0(k+1) = \begin{cases} (1-v)\delta_0(k) + v[n_1(k) - n_1^*] - v\eta, k = k_i \\ \delta_0(k) + v[n_1(k) - n_1^*], \\ \quad k = k_i + 1, \cdots, k_{i+1} - 1 \end{cases}.$$

Hence,

$$\begin{aligned} \delta_0(k_{i+1}) &= \delta_0(k_i + 1) + \sum_{k_i+1}^{k_{i+1}-1} v[n_1(k) - n_1^*] \\ &= (1-v)\delta_0(k_i) + v[n_1(k_i) - n_1^*] - v\eta \\ &+ \sum_{k_i+1}^{k_{i+1}-1} v[n_1(k) - n_1^*] \\ &= (1-v)\delta_0(k_i) - v\eta + \sum_{k_i}^{k_{i+1}-1} v[n_1(k) - n_1^*]. \end{aligned}$$

There are two cases to consider.

Case 1. K is an infinite set. From (34) there then exists $I < \infty$ such that

$$\begin{aligned} \delta_0(k_{i+1}) &= (1-v)\delta_0(k_i) - v\eta + \sum_{k_i}^{k_{i+1}-1} v[n_1(k) - n_1^*] \\ &< (1-v)\delta_0(k_i) - \frac{1}{2}v\eta, \text{ for } i > I. \end{aligned}$$

But then $\delta_0(k) + \eta < 0$ for $k > k_I$, so (35) simplifies to

$$\delta_0(k+1) = (1-v)\delta_0(k) + v[n_1(k) - n_1^*] - v\eta, \quad k > k_I,$$

from which we can conclude that $\delta_0(k)$ converges and

$$\lim_{k\to\infty} \delta_0(k) = -\eta = -[1-2\epsilon]w^{-1}r,$$

so that

$$\begin{aligned} \lim_{k\to\infty} n_0(k) &= n_0^* + \lim_{k\to\infty} \delta_0(k) = n_c + [1-2\epsilon]w^{-1}r \\ &- [1-2\epsilon]w^{-1}r = n_c. \qquad (36) \end{aligned}$$

Case 2. K is a finite set. But then (35) simplifies to

$$\delta_0(k+1) = \delta_0(k) + v[n_1(k) - n_1^*], \quad k > k_I,$$

so that $\delta_0(k)$ again converges, but the limit is

$$\lim_{k\to\infty} \delta_0(k) = \delta_0(k_I) + \sum_{k_I}^{\infty} v[n_1(k) - n_1^*],$$

and

$$\lim_{k\to\infty} n_0(k) = n_0^* + \delta_0(k_I) + \sum_{k_I}^{\infty} v[n_1(k) - n_1^*]. \qquad (37)$$

Lemmas 4.1 and 4.2 together yield the next result.

Theorem 4.1: The freeway of the Example has at least two sets of equilibria: (1) the asymptotically stable equilibrium n^e given by (19), in which every section is congested; and (2) the stable equilibrium of the form n^* given by (24), in which every section, except possibly section 0, is uncongested.

We now find *all* the equilibria for the Example. Let $n = (n_0, \cdots, n_{N-1})$ be any equilibrium, which means

$$\begin{aligned} f_0 &= F = \min\{vn_0, F\}, & (38)\\ f_i &= F - r = \min\{vn_i, F - w[n_{i-1} - n_c], F\},\\ &\quad 1 \le i \le N-1. & (39)\end{aligned}$$

Fact 4.1: Suppose section I is congested for some $I > 0$. Then sections $0, \cdots, I-1$ are also congested, and $n_i = n_i^e = n_c + w^{-1}r$, $0 \le i \le I-1$.

Proof. Suppose section $i \ge 1$ is congested, i.e. $n_i > n_c$. Then, from (39), $f_i = F - r = F - w[n_{i-1} - n_c]$ and so $n_{i-1} = n_c + w^{-1}r = n_{i-1}^e$, and section $i-1$ is also congested. □

Theorem 4.2: n is an equilibrium of the Example if and only if there exists $I \ge 0$ such that sections $i = 0, \cdots, I$ are congested and sections $I+1, \cdots, N-1$ are uncongested, and (see Figure 7)

$$\begin{aligned} n_i &= n_i^e = n_c + w^{-1}r, \quad 0 \le i \le I-1, & (40)\\ n_i &= n_i^* = v^{-1}(F-r), \quad I+2 \le i \le N-1, & (41)\\ n_I &= n_I^e = n_c + w^{-1}r\\ &\quad \text{and } n_{I+1} \in [v^{-1}(F-r), n_c], \text{ or} & (42)\\ n_I &\in [n_c, n_c + w^{-1}r]\\ &\quad \text{and } n_{I+1} = n_i^* = v^{-1}(F-r). & (43)\end{aligned}$$

All these equilibria are stable.

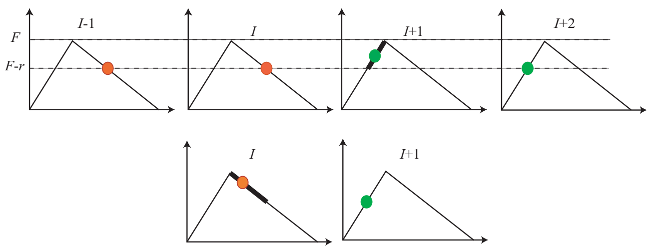

Fig. 7. Equilibrium satisfying (40), (41) and (42) (above), and (43) (below).

Proof. The existence of I and (40) follow from Fact 4.1. Since sections $I+1, I+2, \cdots$ are uncongested, $n_i \le n_c, i \ge I+1$ and so, from (39), for $i - 1 \ge I+1$,

$$f_i = F - r = \min\{vn_i, F - w[n_{i-1} - n_c], F\} = vn_i,$$

which proves (41). Lastly,

$$f_{I+1} = F - r = \min\{vn_{I+1}, F - w[n_I - n_c], F\}.$$

Hence, either $F - r = F - w[n_I - n_c]$, in which case $n_I = n_c + w^{-1}r$ and $n_{I+1} \in [v^{-1}(F-r), n_c]$, or $F - r = vn_{I+1}$, in which case $n_{I+1} = v^{-1}(F-r)$ and $n_I \in [n_c, n_c + w^{-1}r]$.

We now prove stability. The case $I = N-1$ corresponds to the equilibrium n^e, and the case $I = 0$ corresponds to n^*. The two cases are covered by Theorem 4.1.

Now consider an equilibrium $n = (n_0, \cdots, n_{N-1})$ in which sections $0, \cdots, I$ are congested and sections $I+1, \cdots, N-1$ are uncongested, for some $0 < I < N-1$. Then (17), (18) become

$$\begin{aligned} f_0(k) &= \min\{vn_0(k), F\} = F,\\ f_i(k) &= \min\{vn_i(k), F - w[n_{i-1}(k) - n_c], F\}\\ &= F - w[n_{i-1}(k) - n_c], \quad 1 \le i \le I,\\ f_{I+1}(k) &= \min\{vn_{I+1}(k), F - w[n_I(k) - n_c], F\},\\ f_i(k) &= \min\{vn_i(k), F - w[n_{i-1}(k) - n_c], F\}\\ &= vn_i(k), \quad I+2 \le i \le N-1.\end{aligned}$$

Substitution into (14)-(15) give

$$\begin{aligned} [n_0(k+1) - n_c] &= (1-w)[n_0(k) - n_c] + r & (44)\\ [n_i(k+1) - n_c] &= (1-w)[n_i(k) - n_c] + w[n_{i-1} - n_c],\\ &\quad 1 \le i \le I-1, & (45)\\ n_i(k+1) &= n_i(k) - f_i(k) + f_{i+1}(k),\\ &\quad i = I, I+1, & (46)\\ n_i(k+1) &= (1-v)n_i(k) + vn_{i+1}(k),\\ &\quad I+2 \le i \le N-1, & (47)\\ n_N(k+1) &= (1-v)n_N(k) + F - r. & (48)\end{aligned}$$

There are two cases to consider: either (42) holds or (43) holds.

Case 1. Suppose (42) holds. Then

$$\begin{aligned} f_I(k) &= F - w[n_{I-1}(k) - n_c],\\ f_{I+1}(k) &= \min\{vn_{I+1}(k), F - w[n_I(k) - n_c], F\}\\ &= F - w[n_I(k) - n_c].\end{aligned}$$

Substitution into (46) for $i = I$ gives

$$[n_I(k+1) - n_c] = (1-w)[n_I(k) - n_c] + w[n_{I-1} - n_c]. \quad (49)$$

Equations (44), (45), (49) are identical in form to (21), whereas equations (47)-(48) are identical in form to (31)-(32). So we may again conclude that if $|n(0) - n|$ is sufficiently small, $n_i(k)$ converges exponentially for $i \ne I+1$,

$$\begin{aligned} \lim_{k\to\infty} n_i(k) &= n_i^e = n_c + w^{-1}r, 0 \le i \le I & (50)\\ &= n_i^* = v^{-1}(F-r),\\ &\quad I+2 \le i \le N-1; & (51)\end{aligned}$$

and, in particular,

$$\sum_{k=0}^{\infty} |n_I(k) - n_c - w^{-1}r|, \text{ and } \sum_{k=0}^{\infty} |vn_{I+2} - (F-r)|$$

can be made arbitrarily small.

To evaluate the behavior of $n_{I+1}(k)$, we consider (46) for $i = I+1$,

$$\begin{aligned} n_{I+1}(k+1) &= n_{I+1}(k) - f_{I+1}(k) + f_{I+2}(k)\\ &= n_{I+1}(k) - F + w[n_I(k) - n_c] + vn_{I+2}(k)\\ &= n_{I+1}(k) + w[n_I(k) - n_c - w^{-1}r]\\ &+ [vn_{I+2}(k) - (F-r)].\end{aligned}$$

Hence,

$$n_{I+1}(k) - n_{I+1}(0) \;\to\; \sum_{j=0}^{\infty} w[n_I(j) - n_c - w^{-1}r] + \sum_{j=0}^{\infty}[vn_{I+2}(j) - (F - r)],$$

which can be made arbitrarily small. This proves stability of the equilibrium (40), (41), (42).

Case 2. Suppose (43) holds. Then

$$\begin{aligned} f_I(k) &= \min\{vn_I(k), F - w[n_{I-1}(k) - n_c], F\} \\ &= F - w[n_{I-1}(k) - n_c] \\ f_{I+1}(k) &= vn_{I+1}(k). \end{aligned}$$

Substitution into (46) for $i = I + 1$ gives

$$n_{I+1}(k) = (1 - v)n_{I+1}(k) + vn_{I+2}(k). \quad (52)$$

Equations (44), (45) are identical in form to (21), whereas equations (47)-(52) and (48) are identical in form to (31)-(32). So we may again conclude that if $|n(0) - n|$ is sufficiently small, $n_i(k)$ converges exponentially for $i \neq I$,

$$\begin{aligned} \lim_{k\to\infty} n_i(k) &= n_i^e = n_c + w^{-1}r, \quad 0 \le i \le I - 1 \quad (53) \\ &= n_i^* = v^{-1}(F - r), \\ &\qquad I + 1 \le i \le N - 1; \quad (54) \end{aligned}$$

and, in particular,

$$\sum_{k=0}^{\infty} |n_{I-1}(k) - n_c - w^{-1}r|, \text{ and } \sum_{k=0}^{\infty} |vn_{I+1} - (F - r)|$$

can be made arbitrarily small.

To evaluate the behavior of $n_I(k)$ we consider (46) for $i = I$,

$$n_I(k + 1) = n_I(k) - F + w[n_{I-1}(k) - n_c] + vn_{I+1}(k).$$

Similarly to case 1,

$$n_I(k) - n_I(0) \;\to\; \sum_{j=0}^{\infty} w[n_{I-1}(j) - n_c - w^{-1}r] + \sum_{j=0}^{\infty}[vn_{I+1}(j) - (F - r)],$$

which can be made arbitrarily small. This proves stability of the equilibrium (40), (41), (43). □

Figure 8 illustrates how a freeway may exhibit two bottlenecks caused by two on-ramps, each of which causes a congestion pattern according to Theorem 4.2.

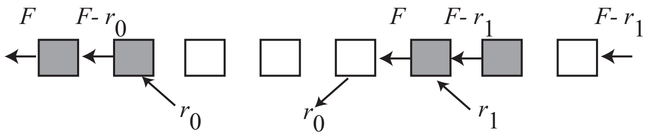

Fig. 8. A freeway with two bottlenecks

V. Empirical Study

CTMsim is a package that simulates the model described above. The model of a part of I-210W form postmiles 25 to 39 in Los Angeles, divided into 40 sections, was calibrated with data obtained from [9]. Figure 9 is a screen shot of the resulting simulation at 7:00 AM (400 minutes past midnight).

The top panel shows capacity (red line) and the flows in each section. The middle panel shows the critical density (red line) and the density in each section. The two large circles indicate the two bottlenecks where congestion begins. The congestion travels upstream, creating two patterns of congestion as predicted by Theorem 4.2, similar to Figure 8.

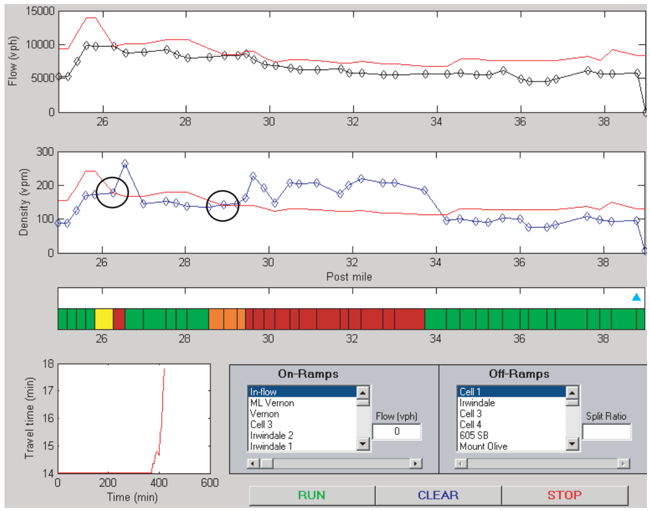

Fig. 9. A freeway with two bottlenecks

VI. Conclusions and Discussion

The paper analyzed the equilibrium behavior of a simplified Asymmetric Cell Transmission Model (ACTM) of freeway traffic when the demand profile is stationary. There are then several equilibria, all of which yield the same flow on every section. There is a unique equilibrium in which every section is uncongested; all other equilibria have some congested sections.

Therefore a congested freeway does not imply that demand exceeds capacity. On the contrary, by appropriate ramp metering control, one can support the same demand while maintaining the freeway uncongested. Indeed, if ramp metering keeps the demand profile strictly feasible, only the uncongested equilibrium is preserved.

Of course effective ramp metering requires adequate storage on the ramp. If that is not available, traffic will spill back into arterial streets. A common practice is to implement a queue override scheme which increases the on ramp flow, causing congestion on the freeway. In effect such an override scheme uses the freeway as a storage area. However, this is frequently a poor decision.

Consider a four-lane freeway with a capacity of 2,000 vehicles per hour per lane, with a fundamental diagram given in the numerical example of Figure 5. The critical density is 40 vehicles per mile. By storing an additional 44 vehicles per lane per mile, or 176 vehicles per freeway mile, the flow is reduced to 1,200 vehicles per hour per lane—an effective reduction in capacity of 40 percent. Moreover, the speed is reduced from 60 mph to under 10 mph. Thus freeways form very inefficient parking areas. It is much more cost-effective to increase the ramp storage capacity.

VII. Acknowledgments

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